EXERCISES 3

1. Find the upper and lower Darboux integrals for the following functions, and determine whether they are integrable or not.

(a) $f(x) = x^3, \ [0,x]$

(b) $f(x) = \begin{cases} x & for\ rational\ x\ in\ [0,b] \\ 0 & for\ irrational\ x\ in\ [0,b] \end{cases}$

(c) $f(x) = \begin{cases} x^2 & for\ rational\ x\ in\ [0,b] \\ 0 & for\ irrational\ x\ in\ [0,b] \end{cases}$

2. Is there a function f on [0,1] that is not integrable for which $|f|$ is integrable?

3. Show that $\left|\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)\right| \leq \frac{16\pi^3}{3}$.

4. Let f be a bounded function on $[a,b]$. Show that if f is integrable on $[a,b]$, then f^2 is integrable on $[a,b]$.

5. Let f and g be integrable functions on $[a,b]$.

(a) Show that fg is integrable on $[a,b]$. (**Hint:** Use $4fg = (f+g)^2 + (f-g)^2$, and Exercise 4).

(b) Show that $\min(f,g)$ and $\max(f,g)$ are integrable on $[a,b]$. (**Hint:** Use $\min(f,g) = \frac{(f+g)-|f-g|}{2}$ and $\max(f,g) = \frac{(f+g)+|f-g|}{2}$).

6. Suppose f and g are continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists x in $[a,b]$ such that $f(x) = g(x)$.

7. Suppose f and g are continuous functions on $[a,b]$ and that $g(x) \geq 0$ for all $x \in [a,b]$. Prove that there exists x in $[a,b]$ such that

$$\int_a^b f(t)g(t)dt = f(x)\int_a^b g(t)dt.$$

8. Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-1}^{x+1} f(t)dt \quad for\ x \in \mathbb{R}.$$

Show that F is differentiable on $\mathbb{R}$ and compute F'.

9. Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_0^{\sin x} f(t)dt \quad for\ x \in \mathbb{R}.$$

Show that F is differentiable on $\mathbb{R}$ and compute F'.

10. Suppose that f is a continuous function on $[a, b]$ and that $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x)dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

11. Suppose that if f is a continuous function on $[a, b]$ satisfying $\int_a^b f(x)g(x)dx = 0$ for every continuous function g on $[a, b]$, then $f(x) = 0$ for all $x \in [a, b]$.

Calculate the following integrals using by the integration methods.

12. $\int \left(3x^4 + 7x^3 + 4x^2 - 8x - 2\right)dx$

13. $\int \frac{\sqrt[4]{x^3} + \sqrt[5]{x} + 3}{\sqrt[3]{x^2}}dx$

14. $\int_0^{\pi} \sin^2 x dx$

15. $\int \cos^2 x dx$

16. $\int \frac{dx}{4x}$

17. $\int 7^x dx$

18. $\int \left(\frac{4}{3^x} + 3^x\right)dx$

19. $\int_{-3}^{0} \frac{5x+3}{x+4}dx$

20. $\int \cos\frac{x}{3}dx$

21. $\int e^{5x}dx$

22. $\int \tan\frac{5x}{2}dx = ?$

23. $\int xe^{x^2}dx$

24. $\int \left(x^2 + 1\right)\left(x^3 + 3x\right)^4 dx$

25. $\int_0^{\pi/2} (\sin x)e^{\cos x}dx$

26. $\int 6^{5x+3}dx$

27. $\int \frac{3^{x+2} + 2^{x+1}}{6^{x-2}}dx$

28. $\int \tan^2 x dx$

29. $\int_0^{\pi/4} \sin^2(2x)dx$

30. $\int \frac{x}{\sin^2\left(x^2\right)}dx$

31. $\int \frac{x^3}{\cos^2\left(x^4\right)}dx$

32. $\int \frac{xdx}{\sqrt{9-4x^4}}$

33. $\int \frac{dx}{x^2\left(\ln x^3\right)^3}$

34. $\int \frac{\cos^2\left(\ln x^5\right)}{x}dx$

35. $\int_0^1 \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$

36. $\int \left(\frac{2}{5^x}+3^x\right)dx$

37. $\int x^5 e^{x^6}dx$

38. $\int \frac{dx}{9x^2-4}$

39. $\int \frac{dx}{\sqrt{16-5x^2}}$

40. $\int_{-1}^{1} \frac{dx}{x^2-4}$

41. $\int \frac{dx}{25x^2-16}$

42. $\int \frac{dx}{x^2+36}$

43. $\int \frac{dx}{4x^2+5}$

44. $\int_{-1}^{1} \frac{dx}{\sqrt{4-x^2}}$

45. $\int \frac{dx}{\sqrt{7-3x^2}}$

46. $\int_0^1 \frac{5x-1}{9x^2+16}dx$

47. $\int \frac{2x+3}{\sqrt{1-x^2}}dx$

48. $\int \frac{\tan\left(\ln x\right)}{x}dx$

49. $\int x^2\left(x^3+4\right)^5 dx$

50. $\int_{-1}^{1} \frac{x}{\sqrt{36-4x^2}}dx$

51. $\int \sin\left(3x\right)e^{\cos(3x)}dx$

52. $\int_0^2 \frac{2^x}{1+4^x}dx$

53. $\int \frac{5^x}{\sqrt{16-25^x}}dx$

54. $\int \left(x^4-3x^2+x\right)^{3/4}\left(4x^3-6x+1\right)dx$

55. $\int \left(\sin\left(5x\right)-\cos\frac{x}{2}\right)dx$

56. $\int \left(3e^{9x}-\cot\frac{x}{9}\right)dx$

57. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{4x}}dx$

58. $\int \left(e^{2x}-1\right)^6 e^{2x}dx$

59. $\int \frac{2x-1}{\sin^2\left(x^2-x\right)}dx$

60. $\int \frac{\sqrt[3]{\arctan x}}{x^2+1}dx$

61. $\int \frac{\ln\left(3x\right)}{2x}dx$

62. $\int xe^{3x}dx$

63. $\int_1^3 x\ln\left(\frac{x}{3}\right)dx$

64. $\int x\sin(4x)\,dx$

65. $\int x^4\ln(2x)\,dx$

66. $\int (4x-3)\ln(5x)\,dx$

67. $\int (3x-2)e^{4x}dx$

68. $\int \arcsin x dx$

69. $\int_0^{1/2} \arctan(2x)\,dx$

70. $\int (2x+5)3^{-x}\,dx$

71. $\int (5x-1)\sin(2x)\,dx$

72. $\int e^x\ln\left(2-e^{-x}\right)dx$

73. $\int \left(x^2+3x-2\right)\sin(5x)\,dx$

74. $\int \left(x^2-2x+5\right)\cos(2x)\,dx$

75. $\int_0^1 x^2\arctan x dx$

76. $\int \frac{x}{\sin^2(x)}dx$

77. $\int \left(x^2-2x\right)e^{5x}dx$

78. $\int x^2\cos(3x)\,dx$

79. $\int e^{3x}\sin(2x)\,dx$

80. $\int_0^{\pi/2} e^{2x}\cos x dx$

81. $\int \sin(\ln x)\,dx$

82. $\int \frac{x+1}{x^2+2x-3}dx$

83. $\int_{-4}^{1} \frac{x^2+2}{x^2+3x-10}dx$

84. $\int \frac{x^2+5}{(x+1)(x-1)^2}dx$

85. $\int_{-1}^{0} \frac{3x^2-2x+5}{(x-1)^2(x+2)^2}dx$

86. $\int \frac{dx}{\left(x^2+1\right)\left(x^2+4\right)}$

87. $\int \frac{dx}{(x-4)\left(x^2+9\right)}$

88. $\int \frac{x^4+1}{x\left(x^2+1\right)^2}dx$

89. $\int \frac{x^3}{\left(x^2+2\right)^2}dx$

90. $\int \sin^3 x\cos^5 x dx$

91. $\int \sin^2 x\cos^3 x dx$

92. $\int \sin^2 3x\cos^4 3x dx$

93. $\int_0^{\pi/2} \cos^4 2x dx$

94. $\int \sin^{2/3} x\cos^3 x dx$

95. $\int \frac{dx}{\sin^3 x\cos^4 x}$

96. $\int \sin\frac{x}{2}\cos\frac{5x}{2}dx$

97. $\int \sin 3x\sin 5x dx$

98. $\int \cos 4x\cos 5x dx$

99. $\int \frac{dx}{\sin^5 x}$

100. $\int \frac{dx}{\sqrt[3]{\sin x \cos^5 x}}$

101. $\int_0^{\pi/4} \frac{dx}{\cos^4 x}$

102. $\int \frac{dx}{\sin^3 x \cos^5 x}$

103. $\int \tan^4 3x dx$

104. $\int \cot^3 x dx$

105. $\int \sinh^5 x \cosh^3 x dx$

106. $\int \sinh^2 x \cosh^3 x dx$

107. $\int \sinh^4 x \cosh^2 x dx$

108. $\int_0^1 \sinh 3x \cosh 2x dx$

109. $\int \sinh x \sinh 4x dx$

110. $\int \cosh 6x \cosh 4x dx$

111. $\int \frac{dx}{\sinh^2 x \cosh x}$

112. $\int \frac{dx}{\sinh^2 x + 4\cosh^2 x}$

113. $\int \tanh^4 x dx$

114. $\int_{-1}^{0} \frac{\sqrt{x+1}+1}{\sqrt[3]{x+1}-1} dx$

115. $\int \frac{dx}{\sqrt[3]{(x-2)^2} + \sqrt[4]{(x-2)^3}}$

116. $\int \frac{dx}{x + \sqrt{x^2+2x+5}}$

117. $\int \frac{1}{\sqrt{-x^2+x+1}} dx$

118. $\int \frac{dx}{\sqrt{-x^2+3x-2}}$

119. $\int \sqrt{-x^2+6x-5} dx$

120. $\int \sqrt{x^2-4x+13} dx$

121. $\int \sqrt{x^2+2x-3} dx$

122. $\int x^7 \sqrt{1+x^4} dx$

123. $\int_1^2 \frac{dx}{x^3 (2+x^5)^{3/5}}$

124. $\int \frac{x^3}{\sqrt{x^2-8x+25}} dx$

125. Evaluate the following integrals by calculating the upper and lower Darboux integrals.

(a) $\int_1^3 (3x+2) dx$

(b) $\int_1^2 (x^2 - 3x + 1) dx$

(c) $\int_0^1 (x - 2x^3) dx$

(d) $\int_0^2 (x^3 - 4x^2) dx$

126. Find the area of the region enclosed by the following curves.

(a) $y = x + 1$, $y = 9 - x^2$, $x = -1$, $x = 2$

(b) $y = \sin x$, $y = e^x$, $x = 0$, $x = \pi/2$

(c) $y = 1 + \sqrt{x}$, $y = \frac{x+3}{3}$

(d) $y = \cos x$, $y = \sec^2 x$, $x = -\pi/4$, $x = \pi/4$

(e) $y = x^2$, $y = \frac{2}{x^2+1}$

(f) $y = |x|$, $y = x^2 - 2$

(g) $y = x^2$, the tangent line to the curve $y = x^2$ at the point (1,1), and the x-axis.