EXERCISES 3

1. Determine in each case whether the given sequences has a limit or not. If it does, prove that your stated value is a limit, and if it does not explain why?

(a) $x_n = \frac{(-1)^n}{n}$

(b) $x_n = (-1)^n\left(1 - \frac{1}{n}\right)$

(c) $x_n = \frac{1+(-1)^n}{n}$

(d) $x_n = (-1)^n n$

(e) $x_n = \sin\left(\frac{n\pi}{2}\right) + \cos n\pi$

(f) $x_n = \frac{3n+1}{5n-2}$

(g) $x_n = \frac{n}{n^2+1}$

(h) $x_n = \frac{n^2}{n^2+1}$

(i) $x_n = \frac{n^3}{n^2+1}$

(j) $x_n = \frac{n^2-n}{n^3+1}$

(k) $x_n = \frac{1}{n}\sin n$

(l) $x_n = \cos\left(\frac{n\pi}{4}\right)$

(m) $x_n = \sin n\pi$

(n) $x_n = \sqrt{n^2+n} - n$

2. Prove that convergence of $\{x_n\}$ implies convergence of $\{|x_n|\}$. Is the converse true?

3. Prove that if $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n > -\infty$, then $\lim_{n\to\infty}(x_n + y_n) = +\infty$.

4. (a) Prove that if $\lim_{n\to\infty} x_n = +\infty$ and $c > 0$, then $\lim_{n\to\infty}(cx_n) = +\infty$.

(b) Prove that if $\lim_{n\to\infty} x_n = +\infty$ and $c < 0$, then $\lim_{n\to\infty}(cx_n) = -\infty$.

5. Suppose that there exists a natural number N_0 such that $x_n \leq y_n$ for all $n \geq N_0$

(a) Prove that if $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} y_n = +\infty$.

(b) Prove that if $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.

6. Show that

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & if\ |a| < 1 \\ 1 & if\ a = 1 \\ +\infty & if\ a > 1 \\ does\ not\ exist & if\ a \leq -1 \end{cases}$$

7. Let $x_1 = 1$ and for $n \geq 1$ let $x_{n+1} = \sqrt{x_n + 1}$. Show that $\lim_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}$.

8. Prove the following limits:

(a) $\lim_{n\to\infty} \frac{n^4+8n}{n^2+9} = +\infty$

(b) $\lim_{n\to\infty} \left(\frac{2^n}{n^2} + (-1)^n\right) = +\infty$

(c) $\lim_{n\to\infty} \frac{a^n}{n!} = 0$

(d) $\lim_{n\to\infty} \left(\frac{3^n}{n^3} - \frac{3^n}{n!}\right) = +\infty$

9. Determine in each case whether the given sequences is monotonic or not.

(a) $x_n = \frac{3}{2n-1}$

(b) $x_n = \frac{(-1)^n}{n^3}$

(c) $x_n = 5n^2$

(d) $x_n = \frac{n+1}{2^n}$

(e) $x_n = \cos\frac{n\pi}{6}$

(f) $x_n = (-3)^n$

10. Let $\{x_n\}$ be a sequence and let $\lim_{n\to\infty} x_n = x$. Prove that $\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = x$.

11. Let $\{x_n\}$ be a sequence of positive numbers and let $\lim_{n\to\infty} x_n = x \neq 0$. Prove that $\lim_{n\to\infty} \sqrt[n]{x_1.x_2 \ldots x_n} = x$ (Hint: Use logarithmic function in Exercise 10).

12. Let $x_1 = 1$ and $x_{n+1} = \frac{1}{5}(x_n + 3)$.

(a) Show that $x_n > \frac{3}{4}$ for all $n \in \mathbb{N}$.

(b) Show that $\{x_n\}$ is a monotonically decreasing sequence.

(c) Prove that $\lim_{n\to\infty} x_n$ exists and its value is $\frac{3}{4}$.

13. For each sequence below, **i)** find its set of subsequential limits, **ii)** find its $\underline{lim} x_n$ and $\overline{lim}\, x_n$.

(a) $x_n = \frac{(-1)^n}{n^2+3}$ **(b)** $x_n = \frac{n+1}{3n-1}$

(c) $x_n = \frac{2}{5n+3}$ **(d)** $x_n = \left(-\frac{1}{7}\right)^n$

(e) $x_n = \sin\frac{n\pi}{6}$ **(f)** $x_n = (-5)^n$

(g) $x_n = \mathrm{n} \cdot \sin\frac{n\pi}{3}$ **(h)** $x_n = (-1)^n - \frac{1}{n}$

14. Prove that $\underline{lim} x_n \leq \overline{lim}\, x_n$.

15. Prove that $\overline{lim}(-x_n) = -\underline{lim} x_n$.

16. Let $\{x_n\}$ and $\{y_n\}$ be sequences such that $x_n \leq y_n$ for all $n \geq N$, where N is a fixed natural number. Show that $\underline{lim} x_n \leq \underline{lim} y_n$ and $\overline{lim}\, x_n \leq \overline{lim}\, y_n$.

17. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences. Show that $\overline{lim}(x_n + y_n) \leq \overline{lim}\, x_n + \overline{lim}\, y_n$ and $\underline{lim}(x_n + y_n) \geq \underline{lim} x_n + \underline{lim} y_n$.

18. Let $\{x_n\}$ be a bounded sequences and let c be a nonnegative real number. Prove that $\underline{lim}(cx_n) = c\underline{lim} x_n$ and $\overline{lim}(cx_n) = c\,\overline{lim}\, x_n$.

19. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences of nonnegative numbers. Prove that $\overline{lim}(x_n y_n) \leq \overline{lim}\, x_n \,.\, \overline{lim}\, y_n$.

20. (a) Prove that if $\lim_{n\to\infty} x_n = +\infty$ and $\underline{lim} y_n > 0$, then $\underline{lim}(x_n y_n) = +\infty$.

(b) Prove that if $\overline{lim}\, x_n = +\infty$ and $\underline{lim} y_n > 0$, then $\overline{lim}(x_n y_n) = +\infty$.