EXERCISES 1

1. Prove that there is no rational number whose squre is 3 or 6.

2. If p is rational ($p \neq 0$) and x is irrational, prove that $p + x$ and px are irrational.

3. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

4. Is the sum of any two irrational numbers always irrational? Why?

5. Is the product of any two irrational numbers always irrational? Why?

6. If a, b, c, d are rational and if x is irratioal, prove that $\frac{ax+b}{cx+d}$ is usually irrational. When do exceptions occur?

7. If $\frac{a}{b} < \frac{c}{d}$ with $b > 0$, $d > 0$, prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

8. Let a and b be positive integers. Prove that $\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$.

9. Prove the following statement for all natural numbers by induction.

(a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

(b) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1))}{2}\right]^2$.

10. Prove that 3 is a factor of $4^n - 1$ for all natural numbers n by induction.

11. Prove $4^{n-1} > n^2$ for $n \geq 3$ by induction.

12. Prove $n^2 < 2^n$ for $n \geq 5$ by induction.

13. Let $\binom{n}{k}$ denote the binomial coefficient,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

where $n \geq 0, k \geq 0$ are integers and $0 \leq k \leq n$. Prove the following assertions by induction.

(a) $\binom{n}{k} = \binom{n}{n-k}$ **(b)** $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ (for $k > 0$)

14. Prove by induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

15. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E.Prove that $\alpha \leq \beta$.

16. Find the sup and inf of each of the following sets of real numbers:

(a) $E = \{x: 3x^2 - 10x + 3 < 0\}$

(b) $E = \{x: (x-a)(x-b)(x-c)(x-d) < 0\}$, where $a < b < c < d$.

(c) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$, where p, q, r take on all positive integer values.

17. Prove that the supremum and the infimum of a set are uniquely determined if they exist.

18. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$.

19. Let A be a nonempty bounded (both below and above) subset of $\mathbb{R}$, and let $\emptyset \neq B \subset A$. Show that $infA \leq infB \leq supB \leq supA$.

20. Let A and B be two sets of positive real numbers bounded above, and let $\alpha = supA$, $\beta = supB$. Let C be set of all products of the form xy, where $x \in A$ and $y \in B$. Prove that $\alpha\beta = supC$.

21. Prove that

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2)$$

if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

22. (Polarization identity) Prove that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \frac{1}{4}(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2)$$

if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$.