Solutions:

1.a) Since $\alpha(t) = (e^t \cos t\,, e^t \sin t, e^t)$ then we have,

$\alpha'(t) = e^t(\cos t\,, \sin t, 1) + e^t(-\sin t\,, \cos t\,, 0) = e^t(\cos t - \sin t\,, \sin t + \cos t\,, 1),$

$\alpha''^{(t)} = e^t(\cos t - \sin t\,, \sin t + \cos t\,, 1) + e^t(-\sin t - \cos t\,, \cos t - \sin t\,, 0)$

$\quad = e^t(-2\sin t\,, 2\cos t\,, 1),$

$\alpha'''^{(t)} = e^t(-2\sin t\,, 2\cos t\,, 1) + e^t(-2\cos t\,, -2\sin t\,, 0)$

$\quad = e^t(-2\sin t - 2\cos t\,, 2\cos t - 2\sin t\,, 1),$

$\|\alpha'(t)\| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1} = e^t\sqrt{3},$

$$\alpha'(t) \times \alpha''(t) = e^{2t}\begin{vmatrix} U_1 & U_2 & U_3 \\ \cos t - \sin t & \sin t + \cos t & 1 \\ -2\sin t & 2\cos t & 1 \end{vmatrix}$$

$\quad = e^{2t}(\sin t - \cos t\,, -\sin t - \cos t\,, 2),$

$\|\alpha'(t) \times \alpha''(t)\| = e^{2t}\sqrt{6},$

$$\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t) = e^{3t}\begin{vmatrix} \cos t - \sin t & \sin t + \cos t & 1 \\ -2\sin t & 2\cos t & 1 \\ -2\sin t - 2\cos t & 2\cos t - 2\sin t & 1 \end{vmatrix}$$

$\quad = e^{3t}[(\cos t - \sin t)(2\sin t) - (\sin t + \cos t)(2\cos t) + 4] = 2e^{3t},$

$T = \frac{\alpha\prime(t)}{\|\alpha\prime(t)\|} = \frac{e^t(\cos t - \sin t, \sin t + \cos t, 1)}{e^t\sqrt{3}} = \frac{1}{\sqrt{3}}(\cos t - \sin t\,, \sin t + \cos t\,, 1),$

$B = \frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|} = \frac{e^{2t}(\sin t - \cos t, -\sin t - \cos t, 2)}{e^{2t}\sqrt{6}}$

$\quad = \frac{1}{\sqrt{6}}(\sin t - \cos t\,, -\sin t - \cos t\,, 2),$

$$N = B \times T = \frac{1}{3\sqrt{2}}\begin{vmatrix} U_1 & U_2 & U_3 \\ \sin t - \cos t & -\sin t - \cos t & 2 \\ \cos t - \sin t & \sin t + \cos t & 1 \end{vmatrix}$$

$\quad = \frac{1}{3\sqrt{2}}(-3\sin t - 3\cos t\,, -3\sin t + 3\cos t\,, 0) = \frac{1}{\sqrt{2}}(-\sin t - \cos t\,, -\sin t + \cos t\,, 0),$

$\kappa = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha\prime(t)\|^3} = \frac{e^{2t}\sqrt{6}}{e^{3t}3\sqrt{3}} = \frac{e^{-t}\sqrt{2}}{3},$

$\tau = \frac{\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2} = \frac{2e^{3t}}{6e^{4t}} = \frac{e^{-t}}{3}.$

b) $s(t) = \int_0^t \|\alpha'(t)\| dt = \int_0^t e^t\sqrt{3} dt = \sqrt{3}(e^t - 1).$

c) Since $\frac{\tau}{\kappa} = \frac{\frac{e^{-t}}{3}}{\frac{e^{-t}\sqrt{2}}{3}} = \frac{1}{\sqrt{2}} \neq 0$ then the curve is a cylindrical helix.

2. Since

$$E_1 = \cos^2 f\, U_1 + \cos f \sin f\, U_2 + \sin f\, U_3$$
$$E_2 = \cos f \sin f\, U_1 + \sin^2 f\, U_2 - \cos f U_3$$
$$E_3 = -\sin f\, U_1 + \cos f\, U_2,$$

then its attitude matrix is

$$A=\begin{pmatrix}\cos^2 f & \cos f\sin f & \sin f\\ \cos f\sin f & \sin^2 f & -\cos f\\ -\sin f & \cos f & 0\end{pmatrix}=\begin{pmatrix}\cos^2 f & \frac{1}{2}\sin 2f & \sin f\\ \frac{1}{2}\sin 2f & \sin^2 f & -\cos f\\ -\sin f & \cos f & 0\end{pmatrix}.$$

Then

$$dA=\begin{pmatrix}-\sin 2f\,df & \cos 2f\,df & \cos f\,df\\ \cos 2f\,df & \sin 2f\,df & \sin f\,df\\ -\cos f\,df & -\sin f\,df & 0\end{pmatrix}.$$

Since

$${}^tA=\begin{pmatrix}\cos^2 f & \dfrac{1}{2}\sin 2f & -\sin f\\ \dfrac{1}{2}\sin 2f & \sin^2 f & \cos f\\ \sin f & -\cos f & 0\end{pmatrix},$$

then we have

$$\omega=dA\ {}^tA=\begin{pmatrix}-\sin 2f\,df & \cos 2f\,df & \cos f\,df\\ \cos 2f\,df & \sin 2f\,df & \sin f\,df\\ -\cos f\,df & -\sin f\,df & 0\end{pmatrix}\begin{pmatrix}\cos^2 f & \dfrac{1}{2}\sin 2f & -\sin f\\ \dfrac{1}{2}\sin 2f & \sin^2 f & \cos f\\ \sin f & -\cos f & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & -df & \cos f\,df\\ df & 0 & \sin f\,df\\ -\cos f\,df & -\sin f\,df & 0\end{pmatrix}$$

b) $\nabla_{fE_2}(fE_1+E_3)=f\nabla_{E_2}(fE_1+E_3)$
$$=f\left(E_2[f]E_1+f\nabla_{E_2}E_1+\nabla_{E_2}E_3\right).$$
Since $E_2[f]=\cos f\sin f\,U_1[f]+\sin^2 f\,U_2[f]-\cos f U_3\,[f]$
$=\frac{1}{2}\sin 2f\frac{\partial f}{\partial x}+\sin^2 f\frac{\partial f}{\partial y}-\cos f\frac{\partial f}{\partial z},$
$\nabla_{E_2}E_1=\sum_j\omega_{1j}(E_2)E_j=\omega_{12}(E_2)E_2+\omega_{13}(E_2)E_3=-df(E_2)E_2+\cos f\,df(E_2)E_3$
$=-E_2[f]E_2+\cos f\,E_2[f]E_3=E_2[f](-E_2+\cos f\,E_3),$
$\nabla_{E_2}E_3=\sum_j\omega_{3j}(E_2)E_j=\omega_{31}(E_2)E_1+\omega_{32}(E_2)E_2=-\cos f\,df(E_2)E_1-\sin f\,df(E_2)E_2$
$=-\cos f\,E_2[f]E_1-\sin f\,E_2[f]E_2=E_2[f](-\cos f\,E_1-\sin f\,E_2),$
then we have
$\nabla_{fE_2}(fE_1+E_3)=f\left(E_2[f]E_1+f\nabla_{E_2}E_1+\nabla_{E_2}E_3\right)$
$=f\left(E_2[f]E_1+fE_2[f](-E_2+\cos f\,E_3)+E_2[f](-\cos f\,E_1-\sin f\,E_2)\right)$
$=fE_2[f]\left((1-\cos f)E_1-(f+\sin f)E_2+f\cos f\,E_3\right)$

c) Since $\theta_i=\sum a_{ij}dx_j$ or $\theta=Ad\xi$, then we have

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}=\begin{pmatrix}\cos^2 f & \frac{1}{2}\sin 2f & \sin f\\ \frac{1}{2}\sin 2f & \sin^2 f & -\cos f\\ -\sin f & \cos f & 0\end{pmatrix}\begin{pmatrix}dx_1\\ dx_2\\ dx_3\end{pmatrix}\ or$$

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}=\begin{pmatrix}\cos^2 f\,dx+\frac{1}{2}\sin 2f\,dy+\sin f\,dz\\ \frac{1}{2}\sin 2f\,dx+\sin^2 f\,dy-\cos f dz\\ -\sin f\,dx+\cos f\,dy\end{pmatrix}.$$

3. Since

$$\begin{aligned}\|F(\boldsymbol{p})-F(\boldsymbol{q})\| &= \|(p_2-2,p_3+5,p_1-1)-(q_2-2,q_3+5,q_1-1)\|\\ &= \|(p_2-q_2,p_3-q_3,p_1-q_1)\| = \|\boldsymbol{p}-\boldsymbol{q}\|\end{aligned}$$

then F is an isometry. Since

$$\mathbf{0}=T^{-1}\big(F(\mathbf{0})\big)=-\boldsymbol{a}+F(\mathbf{0})=-\boldsymbol{a}+(-2,5,-1)$$

then T is a translation by $(-2,5,-1)$. Since

$$\begin{aligned}C(\boldsymbol{p}) &= (T^{-1}F)(\boldsymbol{p})=T^{-1}\big(F(\boldsymbol{p})\big)=F(\boldsymbol{p})+(2,-5,1)\\ &= (p_2-2,p_3+5,p_1-1)+(2,-5,1)=(p_2,p_3,p_1)\end{aligned}$$

then we have

$$\begin{pmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{pmatrix}\begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix}=\begin{pmatrix}p_2\\ p_3\\ p_1\end{pmatrix}\quad\Rightarrow\quad C=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}.$$