EXERCISES 2

1. Find the sum of the following series.

(a) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)}$

(b) $\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2}$

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-n}$

(d) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5^n-1}$

(f) $\sum_{n=1}^{\infty} \ln\left(1 - \frac{1}{(n+1)^2}\right)$

(g) $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

(h) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\ldots(n+r)}$

2. Determine whether the following series converge or diverge, by using the Comparison Test.

(a) $\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

(c) $\sum_{n=1}^{\infty} \frac{n-1}{n^3+3}$

(d) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-n}}$

(e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+n}}$

(f) $\sum_{n=1}^{\infty} \frac{n^2+n}{\sqrt{n^7+1}}$

(g) $\sum_{n=1}^{\infty} 2^n \sin\frac{1}{3^n}$

(h) $\sum_{n=2}^{\infty} \frac{(n^2+1)5^n}{n^{13/4}-n^2}$

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n^{n+1}}}$

(j) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

3. (Limit Comparison Test) Let $\sum a_n$ and $\sum b_n$ be two series with $a_n \geq 0, b_n \geq 0$ for all n, and let $\alpha = \lim_{n\to\infty} \frac{a_n}{b_n}$. Prove the following assertions:

(a) If $0 < \alpha < +\infty$, then either both series converge or both series diverge.

(b) If $\alpha = 0$ and $\sum b_n$ converges, then the series $\sum a_n$ converges.

(c) If $\alpha = +\infty$ and $\sum b_n$ diverges, then the series $\sum a_n$ diverges.

4. Determine whether the following series converge or diverge (for the values of p if p takes part in the terms of series), by using the Limit Comparison Test (Exercise 3).

(a) $\sum_{n=2}^{\infty} \frac{1}{n^p - n^q}$, $(0 < q < p)$

(b) $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n}$, $(0 < q < p)$

(c) $\sum_{n=2}^{\infty} n^p \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right)$

(d) $\sum_{n=1}^{\infty} \frac{3n^2-2}{n^3 3^n}$

(e) $\sum_{n=1}^{\infty} n^3 e^{-\sqrt{n}}$

(f) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt[3]{n^2}}$

(g) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n^2+3n}\right)^{1/3}$

(h) $\sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{\ln n}$

(i) $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n}} \ln\left(\frac{n-1}{n+1}\right)$

(j) $\sum_{n=2}^{\infty} \frac{1}{n} \sin\frac{1}{3n}$

5. Determine whether the following series converge or diverge (for the values of p if p takes part in the terms of series), by using the Cauchy Condensation Test.

(a) $\sum_{n=2}^{\infty} (\ln n)^p$

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$

(d) $\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^p}$

6. Determine whether the following series converge or diverge (for the values of p if p takes part in the terms of series), by using the Root Test.

(a) $\sum_{n=1}^{\infty} n^4 e^{-n}$

(b) $\sum_{n=1}^{\infty} p^n n^p$

(c) $\sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1\right)^n$

(d) $\sum_{n=1}^{\infty} \sqrt{e^n} \left(\frac{2n}{2n-1}\right)^{n^2}$

(e) $\sum_{n=1}^{\infty} 3^{\left(\frac{n^2-n}{3n+1}\right)}$

(f) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^{n^2-n}$

7. Determine whether the following series converge or diverge, by using the Ratio Test.

(a) $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$

(b) $\sum_{n=1}^{\infty} \frac{e^{-n}(3n^2+n)}{2n+3}$

(c) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

(d) $\sum_{n=1}^{\infty} \frac{4.7.10\ldots(3n+1)}{2.4.6\ldots(2n)}$

(e) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{(\ln 2)^n}$

(f) $\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}$

8. (Raabe's Test) Let $\sum a_n$ be a series with $a_n > 0$ for all n, and let $b_n = n\left(1 - \frac{a_{n+1}}{a_n}\right)$. Prove the following assertions:

(a) If $\lim_{n\to\infty} b_n > 1$, then $\sum a_n$ converges.

(b) If $\lim_{n\to\infty} b_n < 1$, then $\sum a_n$ diverges.

9. First show that the Ratio Test gives no information for the following series. Then determine whether they converge or diverge, by using the Raabe's Test.

(a) $\sum_{n=1}^{\infty} \left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\right)^2$

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{4^n (n!)^2}$

10. Show that the following series are convergent by the Alternating Series Test.

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n(3n-1)}{(n-1)\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \ln \frac{n+1}{n}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+1)}{n\ln(n+1)}$

(d) $\sum_{n=2}^{\infty} (-1)^n \sin\left(\frac{1}{\ln n}\right)$

11. Show that the following series are convergent by the Dirichlet's Test.

(a) $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}$ **(b)** $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

(**Hint for (b):** Write $\sin n = \frac{1}{\sin\frac{1}{2}}\left(\sin\frac{1}{2} \cdot \sin n\right)$ and then use the identity $\sin a \cdot \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$.)

12. Show that the following series are convergent by the Abel's Test.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln\left(1+\frac{1}{n}\right)}{n}$ **(b)** $\sum_{n=1}^{\infty} \frac{\left(\frac{\pi}{2} - arc\tan n\right)}{n^2}$

13. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$, if $a_n \geq 0$.

(**Hint:** Use the inequality $\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$ and the fact that $(a_n)^2 < a_n$ for large n.)

14. Prove that the divergence of $\sum a_n$ implies the divergence of $\sum \frac{a_n}{1+a_n}$, if $a_n > 0$.

(**Hint:** Show the divergence for the cases when $\{a_n\}$ is bounded and when $\{a_n\}$ is unbounded, respectively.)

15. Prove that the divergence of $\sum a_n$ implies the divergence of $\sum n a_n$.

(**Hint:** Assume $\sum n a_n$ is convergent and use the Dirichlet's Test for the series $\sum n a_n$ and $\sum \frac{1}{n}$ to reach a contradiction.)

16. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \sqrt{a_n a_{n+1}}$, if $a_n > 0$. Show that the converse is also true if $\{a_n\}$ is monotonic.

(**Hint:** Use the inequality $\sqrt{a_n a_{n+1}} \leq \frac{a_n + a_{n+1}}{2}$ for all n.)

17. Prove that the absolutely convergence of $\sum a_n$ implies the absolutely convergence of the following series:

(a) $\sum (a_n)^2$ **(b)** $\sum \frac{a_n}{1+a_n}, \quad (a_n \neq -1)$ **(c)** $\sum \frac{(a_n)^2}{1+(a_n)^2}$

18. Prove that the series

$$1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \cdots$$

converges non-absolutely (or conditionally), and show that the rearrangement of it such that

$$\left(1 + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{2}}\right) + \left(\frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{7}} - \frac{1}{\sqrt[3]{4}}\right) + \cdots,$$

in which two positive terms are always followed by one negative, diverges.

19. Prove that the series

$$\frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + \cdots$$

diverges, and show that the rearrangement of it such that

$$\left(\frac{4}{3} - \frac{5}{4}\right) + \left(\frac{6}{5} - \frac{7}{6}\right) + \cdots,$$

in which two consecutive terms are grouped by adding paranthesis, converges.

20. Find the interval of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{e^n}{n!} x^n$ **(b)** $\sum_{n=1}^{\infty} \frac{n^2}{5^n} x^n$

(c) $\sum_{n=1}^{\infty} \frac{n!}{3^n} (x-1)^n$ **(d)** $\sum_{n=1}^{\infty} \left(\frac{n-1}{3n+2}\right)^{2n-3} (x+1)^n$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n (n-2)}{n^2 2^n} (x-2)^n$ **(f)** $\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n-3)4^n} (x+2)^n$

(g) $\sum_{n=1}^{\infty} (n+2) \sin\left(\frac{1}{5n-1}\right) x^n$ **(h)** $\sum_{n=1}^{\infty} \frac{1}{4n-3} \left(\frac{x-2}{x+3}\right)^{2n-1}$

21. Find the Taylor series for the following functions about the indicated values of x, and show that whether they can be given by their Taylor series about that values or not.

(a) $f(x) = \sin 2x$, $x = 0$

(b) $f(x) = e^{-3x}$, $x = 1$

(c) $f(x) = \ln(2x + 1)$, $x = 0$

(d) $f(x) = arc\tan x$, $x = 0$

(e) $f(x) = \frac{1}{1-x}$ $x = 0$

(f) $f(x) = \sin^2 x$, $x = \pi$

22. Find the Mclaurin series for $f(x) = \cosh(x)$ and $g(x) = \sinh(x)$, by using the Mclaurin series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

23. Find the Mclaurin series for $f(x) = \frac{1}{1+x^2}$, by using Exercise 21(e).

24. Find the Mclaurin series for $f(x) = \frac{1}{(1-x)^2}$, by using the (Cauchy) product of the series for $\frac{1}{1-x}$ in Exercise 21(e) with itself.