Solutions:

1. a) If $x_n = \sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$, then $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right) = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$. Because, if we choose the natural number $N > \frac{1}{\varepsilon}$, then for all $n \geq N$ we have $\left|x_n - \frac{1}{2}\right| = \left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| = \left|\frac{2-\left(\sqrt{1+\frac{1}{n}}+1\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| = \left|\frac{1-\sqrt{1+\frac{1}{n}}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| = \left|\frac{-\frac{1}{n}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| < \frac{1}{n} < \varepsilon$.

b) If$x_n = \frac{2^n}{n^2} = \frac{(1+1)^n}{n^2} = \frac{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}}{n^2}$, then $\lim_{n\to\infty} x_n = +\infty$. Because, if we choose the natural number $N > 6M+3$, then for all $n \geq N$ we have $x_n = \frac{2^n}{n^2} = \frac{(1+1)^n}{n^2} = \frac{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}}{n^2} > \frac{\binom{n}{3}}{n^2} = \frac{n(n-1)(n-2)}{6n^2} = \frac{n^3-3n^2+2n}{6n^2} > \frac{n}{6} - \frac{1}{2} + \frac{1}{3n} = \frac{n-3}{6} + \frac{1}{3n} > \frac{n-3}{6} > M$.

2. a) We will prove it by induction:

$A(1)$ is true, because $x_2 = \frac{\sqrt{8x_1+1}}{2} = \frac{3}{2} > 1 = x_1$.

Assume that the assertion $A(n)$: $x_{n+1} > x_n$ is true.

Then we have

$$x_{n+2} = \frac{\sqrt{8x_{n+1}+1}}{2} > \frac{\sqrt{8x_n+1}}{2} = x_{n+1}.$$

This means $A(n+1)$ is also true. Then by induction, $x_{n+1} > x_n$ is true for all natural numbers n. So $\{x_n\}$ is monotonically increasing.

b) Since $\{x_n\}$ is monotonically increasing, then $\frac{\sqrt{8x_n+1}}{2} > x_n$ or $4{x_n}^2 - 8x_n - 1 < 0$. So we have $\left(x_n - \frac{2+\sqrt{5}}{2}\right)\left(x_n - \frac{2-\sqrt{5}}{2}\right) < 0$ or $0 < x_n < \frac{2+\sqrt{5}}{2}$, since the terms of $\{x_n\}$ are positive.

c) From a) and b) $\{x_n\}$ is convergent, and say $\lim_{n\to\infty} x_n = l$. Then $\lim_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \frac{\sqrt{8x_n+1}}{2}$ or $l = \frac{\sqrt{8l+1}}{2}$. Then we have $l = \frac{2+\sqrt{5}}{2}$.

3. a) $\lim_{x\to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}$. To show this limit we have to find a $\delta > 0$ such that for every $\varepsilon > 0$, $|f(x) - L| < \varepsilon$ for which $0 < |x - x_0| < \delta$. If we write

$$\left|\frac{x+1}{x^2+x+1} - \frac{2}{3}\right| = \left|\frac{-2x^2+x+1}{3(x^2+x+1)}\right| = \left|\frac{x-x^2+1-x^2}{3(x^2+x+1)}\right| = \left|\frac{(1-x)(1+2x)}{3(x^2+x+1)}\right| < |1-x| < \varepsilon$$

then we can choose $\delta > 0$ such that $|x - 1| < \varepsilon = \delta$.

b) $\lim_{x\to 3} \frac{x}{(x-3)^2} = +\infty$. We need to show that for every real number M there exists $\delta > 0$ such that $f(x) > M$ for which $0 < |x - x_0| < \delta$. If we choose $0 < \delta < 1$ then we have $|x - 3| < \delta < 1$ or $2 < x < 4$. So we get

$$M < \frac{x}{(x-3)^2} < \frac{4}{(x-3)^2}$$

Then we can choose $0 < \delta < 1$ such that $|x - 3| < \frac{2}{\sqrt{M}} = \delta$.

4. We can write

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ \frac{x}{2}, & 0 \le x < 1 \\ \frac{x}{3}, & 1 \le x < 2 \end{cases}$$

i) For the right-hand limit at $x = 0$ we have,

$$\lim_{x\to 0^+} f(x) = \lim_{\substack{h\to 0\\ h>0}} f(0+h) = \lim_{\substack{h\to 0\\ h>0}} \left(\frac{h}{2}\right) = 0.$$

For the left-hand limit at $x = 0$ we have

$$\lim_{x\to 0^-} f(x) = \lim_{\substack{h\to 0\\ h>0}} f(0-h) = \lim_{\substack{h\to 0\\ h>0}} (h) = 0.$$

Since $\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x)$, then the function is continuous at $x = 0$.

ii) For the right-hand limit at $x = 1$ we have,

$$\lim_{x\to 1^+} f(x) = \lim_{\substack{h\to 0\\ h>0}} f(1+h) = \lim_{\substack{h\to 0\\ h>0}} \left(\frac{1+h}{3}\right) = \frac{1}{3}.$$

For the left-hand limit at $x = 1$ we have

$$\lim_{x\to 1^-} f(x) = \lim_{\substack{h\to 0\\ h>0}} f(1-h) = \lim_{\substack{h\to 0\\ h>0}} \left(\frac{1-h}{2}\right) = \frac{1}{2}.$$

Since $\lim_{x\to 1^+} f(x) = \frac{1}{3} \neq \frac{1}{2} = \lim_{x\to 1^-} f(x)$, then $x = 1$ is a discontinuity point of the function. So f has a discontinuity of the first kind at $x = 1$.