EXERCISES 4

1. Find the domains of the following functions.

(a) $f(x) = \sqrt{x+1}$ **(b)** $f(x) = \sqrt[3]{2x+3}$

(c) $f(x) = \frac{1}{4-x^2}$ **(d)** $f(x) = \frac{1}{\sqrt{|x|-x}}$

(e) $f(x) = \frac{1}{3-ln(x+1)}$ **(f)** $f(x) = \log\left(\frac{x^2-1}{x^2+1}\right)$

(g) $f(x) = \sqrt{ln\left(\frac{5x-x^2}{4}\right)}$ **(h)** $f(x) = \log_2(\log_3(\log_4 x))$

(i) $f(x) = \log_{x-2}(-x^2+8x+9)$ **(j)** $f(x) = \log\left(1-\frac{1}{\log(x^2-5x+16)}\right)$

2. Find the domains of $f \circ g$ and $g \circ f$ for the following functions.

(a) $f(x) = \frac{1+x}{1-x}$, $g(x) = 1 + x^2$

(b) $f(x) = x^3 - 1$, $g(x) = \sqrt{x}$

(c) $f(x) = \sqrt{x}$, $g(x) = ln\left(\frac{3x-x^2}{2}\right)$

3. Find the domains of $f \pm g$, fg, f/g and g/f for the following functions.

(a) $f(x) = \sqrt{x^2-3x+2}$, $g(x) = \frac{1}{\sqrt{3+2x-x^2}}$

(b) $f(x) = \frac{2x+1}{|x-2|-1}$, $g(x) = \sqrt{3x^2+x+4}$

(c) $f(x) = \sqrt{11+10x-x^2}$, $g(x) = \frac{1}{\log(x)-1}$

(d) $f(x) = \log(x^2-9x+18)$, $g(x) = \frac{1}{\sqrt{x^2-9}}$

(e) $f(x) = \frac{3}{\sqrt{4-x^2}}$, $g(x) = ln(x^3-x)$

(f) $f(x) = \frac{1}{x^2+1}$, $g(x) = \sqrt{x-\sqrt{x+1}}$

4. Verify the following limits by $\varepsilon - \delta$ property.

(a) $\lim_{x\to 1}(3x-8) = 5$ **(b)** $\lim_{x\to +\infty}\frac{5x+1}{3x+9} = \frac{5}{3}$

(c) $\lim_{x\to 1}\frac{1}{(1-x)^2} = +\infty$

5. Find the the following limits.

(a) $\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x}$ **(b)** $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

(c) $\lim_{x\to 0}\frac{1-\cos x}{x(\sqrt{1+x}-1)}$ **(d)** $\lim_{x\to 1}\frac{\cos\left(\frac{\pi x}{2}\right)}{1-x}$

(e) $\lim_{x\to +\infty}\left(\frac{x^3}{3x^2-4}-\frac{x^2}{3x+2}\right)$ **(f)** $\lim_{x\to 1}\frac{x^n-1}{x^m-1}$ $(n, m \in \mathbb{N})$

6. Find the the following limits at the indicated points.

(a) $f(x) = [\![1-x]\!] + |1-x|$, $x_0 = 0$.

(b) $f(x) = \frac{6+4^{1/(x-1)}}{11+4^{1/(x-1)}}$, $x_0 = 1$.

(c) $f(x) = \frac{[\![x]\!]^2-9}{x^2-9}$, $x_0 = 3$.

(d) $f(x) = \frac{|x-3|+2sgn(x-1)}{[\![2-x]\!]}$, left-hand limit at $x_0 = 1$.

7. Show that $f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0, \end{cases}$ is continuous at $x_0 = 0$ by $\varepsilon - \delta$ property.

8. Decide the continuities of the following functions.

(a) $f(x) = x - [\![x]\!]$, on the set $E = [-2, 2)$.

(b) $f(x) = [\![x]\!][\![x+1]\!]$, at the point $x_0 = 0$.

(c) $f(x) = sgn(\sin x)$, on the set $E = [0, 2\pi]$.

(d) $f(x) = \frac{|x|}{[\![x]\!]+2}$, on the set $E = [-1, 2)$.

(e) $f(x) = \frac{1}{x^2+3}$, on the set $E = \mathbb{R}$.

(f) $f(x) = 2^{1/x}$, at the point $x_0 = 0$.

(g) $f(x) = \begin{cases} \frac{x-1}{3+2^{1/(1-x)}}, & x \neq 1 \\ 0, & x = 1 \end{cases}$, at the point $x_0 = 1$.

(h) $f(x) = \begin{cases} \frac{2x^2+3}{5}, & -\infty < x \leq 1 \\ 6-5x, & 1 < x < 3 \\ x-3, & 3 \leq x < +\infty \end{cases}$, on the set $E = \mathbb{R}$.

(i) $f(x) = \begin{cases} \sin\left(\frac{\pi x}{2}\right), & 0 < x < 1 \\ 1 + lnx, & 1 < x < 2 \\ 2, & x = 1 \end{cases}$, at the point $x_0 = 1$.

(j) $f(x) = \begin{cases} x^2 - 2x, & x < 2 \\ 1, & x = 2 \\ x-2, & x > 2 \end{cases}$, at the point $x_0 = 2$.

(k) $f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1 \\ -x+4, & x > 1 \end{cases}$, at the point $x_0 = 1$.

(l) $f(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ 2, & x = 0 \\ x+1, & x > 0 \end{cases}$, at the point $x_0 = 0$.

9. Find the value of a if the function $f(x) = \begin{cases} 6 \cdot 3^x, & x < 0 \\ 3a + x, & x \geq 0 \end{cases}$ is continuous at $x_0 = 0$.

10. Show that the function $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - \mathbb{Q} \end{cases}$ is discontinuous at every $x \in \mathbb{R}$.

11. Show that the function $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - \mathbb{Q} \end{cases}$ is continuous at $x_0 = 0$, and discontinuous at the other real numbers.

12. Show that if the function $-f$ assumes its maximum at $x_0 \in [a,b]$, then f assumes its minimum at x_0.

13. Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$.

14. Prove that $x = \cos x$ for some $x \in \left(0, \frac{\pi}{2}\right)$.

15. Prove that $x \cdot 2^x = 1$ for some $x \in (0,1)$.

16. Prove that a polynomial function of odd degree has at least one real root.

17. Prove that each of the following functions is uniformly continuous on the indicated set by verifying $\varepsilon - \delta$ property.

(a) $f(x) = 5x - 2$, on the set $E = \mathbb{R}$.

(b) $f(x) = x^2$, on the set $E = [1,2]$.

(c) $f(x) = \frac{1}{x}$, on the set $E = [1, +\infty)$.

(d) $f(x) = \frac{x}{2x+1}$, on the set $E = [0,1]$.

(e) $f(x) = \frac{x-1}{3x-2}$, on the set $E = [2, +\infty)$.

18. (a) Prove that if f is uniformly continuous on a bounded set E, then f is a bounded function on E.

(b) Use (a) to prove that $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0,1)$.