21.10.2018

Ad ve Soyad :
Numara :
Bölüm :

Soru:	1	2	3	Toplam
Türü:	S	S	S	
Puan:				

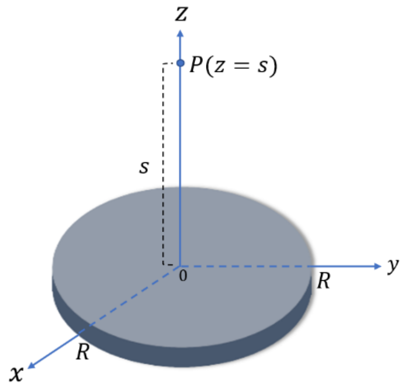
ELEKTROMAGNETiK TEORi

2019-2020 Güz Dönemi Örgün Öğretim 1. Kısa Sınavı

1. Homojen σ yüzeysel yük yoğunluğuna ve R yarıçapına sahip bir disk merkezi orijinde olacak biçimde xy düzlemine şekildeki gibi yerleştirilmiştir. Diskin $P(z=s)$ noktasında meydana getirdiği,

a. Potansiyeli doğrudan doğruya integrasyon ile hesaplayınız. *(50 Puan)*

b. Elektrik alanı hesaplayınız. *(50 Puan)*



2. Bir bölgedeki elektrik alan küresel koordinatlarda, a ve b sabitler olmak üzere aşağıdaki gibi verilmektedir:

$$\vec{E}(r,\theta,\phi) = \frac{a\,\vec{e_r} + b\,sin\theta\,cos\phi\,\vec{e_\phi}}{r}$$

i. Yük yoğunluğu nedir? *(40 Puan)*

ii. a ve b sabitlerinin birimleri SI birim sisteminde nedir? *(30 Puan)*

iii. Doğada böyle bir elektrik alan bulunabilir mi? Yanıtınızı gerekli fiziksel ve matematiksel dayanaklar ile açıklayınız. *(30 Puan)*

3. Sonsuz uzunluktaki, R yarıçapındaki bir silindirin içindeki hacimsel yük yoğunluğu silindirin ekseninden İtibaren $\rho(r) = kr$ şeklinde değişmektedir (k bir sabit). Silindirin içindeki ve dışındaki elektrik alanı bulunuz. *(100 puan)*

Not: Yalnızca 1., 2. ve 3. sorulardan biri seçmelidir. Sınav süresi 30 dakikadır!

Başarılar.
Doç. Dr. Ertan GÜDEKLİ
Araş. Gör. Çağlar ÇETİNKAYA

Koordinat Sistemi	(u_1, u_2, u_3)	x	y	z	h_i
Kartezyen	(x, y, z)	x	y	z	$1, 1, 1$
Silindirik	(ρ, θ, z)	$\rho\cos\theta$	$\rho\sin\theta$	z	$1, \rho, 1$
Küresel	(r, θ, φ)	$r\sin\theta\cos\varphi$	$r\sin\theta\sin\varphi$	$r\cos\theta$	$1, r, r sin\theta$

Eğrisel koordinatlarda,

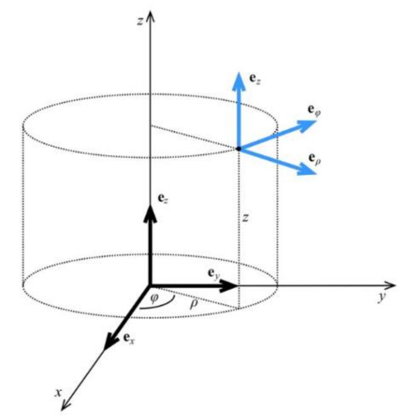
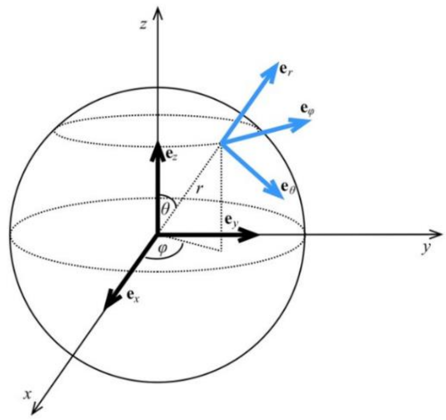
Gradient: $$\vec{\nabla} f = \frac{1}{h_i}\frac{\partial f}{\partial u_i}\vec{e}_i$$

Diverjans: $$\vec{\nabla}.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_1h_3) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]$$

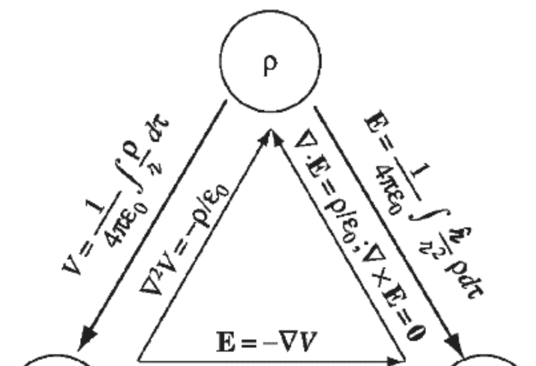
Rotasyonel: $$\vec{\nabla}\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

Laplasyen $$\nabla^2 f = \vec{\nabla}.(\vec{\nabla} f) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial A_1}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial A_2}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial A_3}{\partial u_3}\right)\right]$$

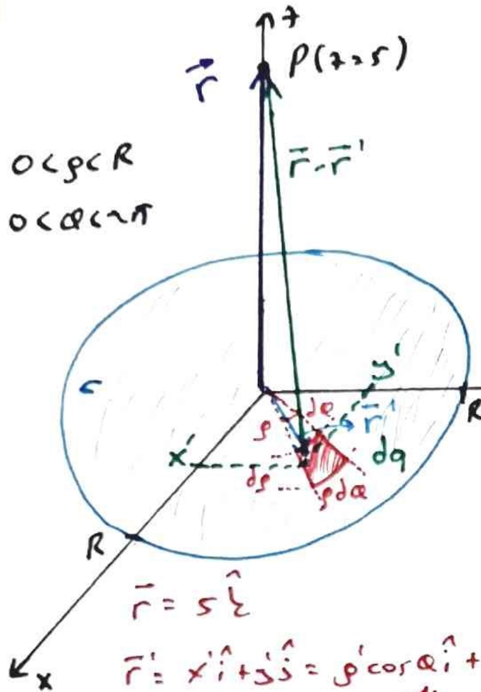
Laplasyen $$\nabla^2\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

Silindirik koordinatlarda taban vöktörlerinin yönelimi:	Küresel koordinatlarda taban vöktörlerinin yönelimi:
	

Silindirik koordinatlarda taban dönüşümü:	Küresel koordinatlarda taban dönüşümü:
$\widehat{e_\rho} = \cos\theta\,\hat{\imath} + sin\theta\,\hat{\jmath}$	$\widehat{e_r} = \sin\theta\cos\varphi\,\hat{\imath} + cos\theta\cos\varphi\,\hat{\jmath} + \cos\theta\,\hat{k}$
$\widehat{e_\theta} = -\sin\theta\,\hat{\imath} + cos\theta\,\hat{\jmath}$	$\widehat{e_\theta} = \cos\theta\cos\emptyset\,\hat{\imath} + cos\theta\sin\varphi\,\hat{\jmath} - sin\theta\,\hat{k}$
$\widehat{e_z} = \hat{k}$	$\widehat{e_\phi} = -\sin\varphi\,\hat{\imath} + cos\theta\hat{\jmath}$



1) a)



$$V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{dq}{|\vec{r}-\vec{r}'|}$$

$$\sigma=\frac{Q}{A}=\frac{dq}{dA}\Rightarrow\sigma=\frac{dq}{dA}$$

$x'=\rho'\cos\varphi'$

$y'=\rho'\sin\varphi'$

$z'=z'$

$dA=dx'dy'=\rho'd\rho'd\varphi'$

$dq=\sigma\, dA$

$dq=\sigma\, dx'dy'$

$\vec{r}=s\hat{z}$

$\vec{r}'=x'\hat{i}+y'\hat{j}=\rho'\cos\varphi'\hat{i}+\rho'\sin\varphi'\hat{j}$

$|\vec{r}-\vec{r}'|=\left[x'^2+y'^2+z^2\right]^{1/2}=\left[\rho'^2+s^2\right]^{1/2}$

$$V(r)=\frac{1}{4\pi\varepsilon_0}\int_A\frac{\sigma\rho'd\rho'd\varphi'}{(\rho'^2+s^2)^{1/2}}=\frac{\sigma}{4\pi\varepsilon_0}\int_{\varphi=0}^{\varphi=2\pi}d\varphi'\int_{\rho=0}^{\rho=R}\frac{\rho'}{(\rho'^2+s^2)^{1/2}}d\rho'$$

$\underbrace{\varphi=0}_{2\pi}$

$\rho'^2+s^2=u$

$du=2\rho'd\rho'$

$d\rho'=\frac{du}{2\rho'}$

$$V(r)=\frac{\sigma}{4\pi\varepsilon_0}(2\pi)\int\frac{\rho'\frac{du}{2\rho'}}{\sqrt{u}}=\frac{\sigma}{4\varepsilon_0}\int_{u=s^2}^{s^2+R^2}u^{-1/2}du=\frac{\sigma}{4\varepsilon_0}\left[\frac{u^{1/2}}{1/2}\right]=\frac{\sigma}{2\varepsilon_0}\left(\sqrt{\rho'^2+s^2}\right)\Big|_{\rho'=0}^{\rho'=R}$$

$$V(r)=\frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+s^2}-|s|\right]\text{//}\Rightarrow V(\underline{z=s})=\frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+s^2}-|s|\right]\text{//}$$

b) Herhangi bir z noktası için;

$$V(z)=\frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-|z|\right],\qquad \vec{E}=-\vec{\nabla}V=-\vec{\nabla}V(z)=-\frac{\partial}{\partial z}V(z)\,\hat{z}$$

$$\Rightarrow\bar{E}(z)=\frac{-\sigma}{2\varepsilon_0}\frac{\partial}{\partial z}\left[(R^2+z^2)^{1/2}-z\right]\hat{k}=\frac{-\sigma}{2\varepsilon_0}\left[\frac{2z}{2}(R^2+z^2)^{-1/2}-1\right]\hat{z}$$

$$\bar{E}(z)=-\frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{R^2+z^2}}-1\right]\hat{z}\Rightarrow\vec{E}(z)=\frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{|z|}-\frac{1}{(R^2+z^2)^{1/2}}\right]\hat{z}\text{//}$$

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2) a) $\oint \bar{E}.d\bar{A} = \frac{Q_{iç}}{\varepsilon_0} \Rightarrow \bar{\nabla}.\bar{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \rho = \varepsilon_0 \bar{\nabla}.\bar{E}$

$$\rho = \varepsilon_0 \left[\frac{1}{h_r h_\theta h_\phi} \left(\frac{\partial}{\partial r}(h_\theta h_\phi E_r) + \frac{\partial}{\partial \theta}(h_r h_\phi E_\theta) + \frac{\partial}{\partial \phi}(h_r h_\theta E_\phi) \right) \right]$$

$$\bar{E} = \underbrace{\frac{a}{r}}_{E_r} \hat{e}_r + \underbrace{\frac{b}{r} \sin\theta \cos\phi}_{E_\phi} \hat{e}_\phi = E_r \hat{e}_r + E_\theta \hat{e}_\theta + E_\phi \hat{e}_\phi$$

$h_r = 1$

$h_\theta = r$

$h_\phi = r\sin\theta$

$E_r = \frac{a}{r}, \quad E_\theta = 0, \quad E_\phi = \frac{b}{r}\sin\theta\cos\phi$

$$\Rightarrow \rho = \varepsilon_0 \left[\underbrace{\frac{1}{r^2\sin\theta}}_{1/h_r h_\theta h_\phi} \frac{\partial}{\partial r}\left(\underbrace{r^2\sin\theta}_{h_\theta h_\phi} \underbrace{\frac{a}{r}}_{E_r} \right) + \underbrace{\frac{1}{r^2\sin\theta}}_{1/h_r h_\theta h_\phi} \frac{\partial}{\partial \theta}\left(\underbrace{r\sin\theta}_{h_r h_\phi} . \underbrace{0}_{E_\theta} \right) + \underbrace{\frac{1}{r^2\sin\theta}}_{1/h_r h_\theta h_\phi} \frac{\partial}{\partial \phi}\left(\underbrace{r}_{h_r h_\theta} \underbrace{\frac{b}{r}\sin\theta\cos\phi}_{E_\phi} \right) \right]$$

$$\Rightarrow \rho = \varepsilon_0 \left[\frac{a}{r^2} + 0 - \frac{b}{r^2}\sin\phi \right] \Rightarrow \rho = \varepsilon_0 \left(\frac{a}{r^2} - \frac{b}{r^2}\sin\phi \right)$$

b) Elektrik alan korunumlu bir vektör alanıdır. Bu yüzden kapalı bir C eğrisi üzerinden integrali sıfırdır;

$$\oint_C \bar{E}.d\bar{l} = 0 \quad \text{veya Stokes Teoremi'nden} \Rightarrow \bar{\nabla}\times\bar{E} = 0 \quad \text{olmalı.}$$

$$\nabla\times\bar{E} = \frac{1}{h_r h_\theta h_\phi} \begin{vmatrix} h_r\hat{e}_r & h_\theta\hat{e}_\theta & h_\phi\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ h_r E_r & h_\theta E_\theta & h_\phi E_\phi \end{vmatrix} = \frac{1}{r^2\sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ E_r & r.0 & r\sin\theta\, E_\phi \end{vmatrix}$$

$$\bar{\nabla}\times\bar{E} = \frac{1}{r^2\sin\theta} \left[\hat{e}_r \left(\frac{\partial}{\partial \theta} r\sin\theta E_\phi - 0 \right) - r\hat{e}_\theta \left(\frac{\partial}{\partial r} r\sin\theta E_\phi - \frac{\partial}{\partial \phi} E_r \right) + r\sin\theta\,\hat{e}_\phi \left(0 - \frac{\partial}{\partial \theta} E_r \right) \right]$$

- $\frac{\partial}{\partial \theta}\left(r\sin\theta \underbrace{\frac{b}{r}\sin\theta\cos\phi}_{E_\phi} \right) = b\cos\phi \frac{\partial}{\partial \theta}\sin^2\theta$

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$$\Rightarrow \frac{\partial}{\partial \theta}\left(r\sin\theta \, \frac{b}{r} \sin\theta \cos\phi \right) = b\cos\phi \, 2\sin\theta \cos\theta$$

- $$\frac{\partial}{\partial r}\left(r\sin\theta \, E_\phi \right) = \frac{\partial}{\partial r}\left(\cancel{r}\sin\theta \, \frac{b}{\cancel{r}} \sin\theta\cos\phi \right) = 0$$

- $$\frac{\partial}{\partial \phi}(E_r) = \frac{\partial}{\partial \phi}\left(\frac{a}{r}\right) = 0 \qquad \bullet \ \frac{\partial}{\partial \theta}(E_r) = \frac{\partial}{\partial \theta}\left(\frac{a}{r}\right) = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = \frac{1}{r^2 \sin\theta}\left[\hat{e}_r \left(b\cos\phi \, 2\sin\theta\cos\theta \right) - r\,\hat{e}_\theta \left(0 - 0 \right) + r\sin\theta \, \hat{e}_\phi \left(0 - 0 \right) \right]$$

$$= 2b \cos\phi\cos\theta \, \hat{e}_r \neq \vec{0}$$ Doğada böyle bir alan bulunamaz!!!

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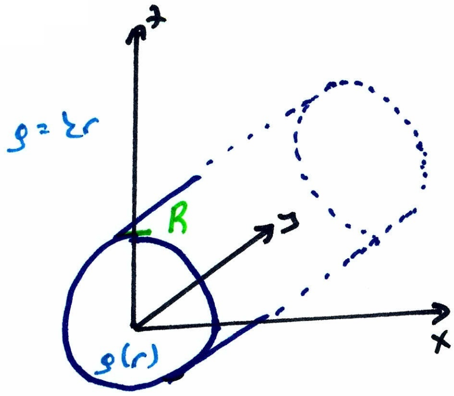
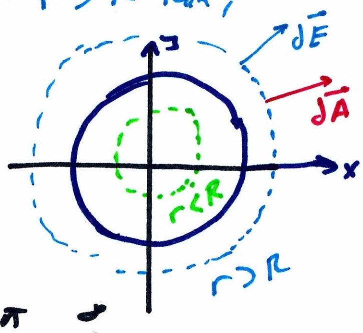
3)



$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \rightarrow \oiint \vec{E}\cdot d\vec{A} = \frac{Q_{iç}}{\varepsilon_0}$$

• $r > R$ için;



$$\int \vec{E}\cdot d\vec{A} = \frac{1}{\varepsilon_0}\int \rho\, dV$$

$dV = r\,dr\,d\varphi\,dz$

$dA = r\,d\varphi\,dz$

$\vec{E} \parallel d\vec{A}, \quad \rho = kr$

$$\Rightarrow \int_{\varphi=0}^{2\pi}\int_{z=0}^{\ell} E\, r\, d\varphi\, \cancel{dz} = \frac{k}{\varepsilon_0}\int_{r=0}^{R} r\, r\, dr \int_{\varphi=0}^{2\pi} d\varphi \cancel{\int_{z=0}^{\ell} dz} \Rightarrow E\, 2\pi r = \frac{k}{\varepsilon_0} 2\pi \left[\frac{r^3}{3}\right]_{r=0}^{R}$$

$$E\, \cancel{2\pi} r = \frac{k\, \cancel{2\pi}}{\varepsilon_0} \frac{R^3}{3} \Rightarrow \boxed{\vec{E} = \frac{k}{3\varepsilon_0}\frac{R^3}{r}\hat{e}_r}$$

• $r < R \Rightarrow$

$$\int E\, dA = \frac{1}{\varepsilon_0}\int \rho(r)\, dv \Rightarrow \int_0 E\, r\, d\varphi \cancel{\int_{z=0} dz} = \frac{k}{\varepsilon_0}\int_0^{r} r^2 dr \int_0^{2\pi} d\varphi \cancel{\int dz}$$

$$\Rightarrow E\, \cancel{2\pi r} = \frac{k}{\varepsilon_0}\cancel{2\pi}\frac{r^{\cancel{3}}}{3} \Rightarrow \boxed{\vec{E} = \frac{k}{3\varepsilon_0} r^2 \hat{e}_r}$$

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