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Ad ve Soyad :
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Bölüm :

Soru:	1	2	3	4	Toplam
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Fizikte Matematik Metodlar

2018-2019 Güz Dönemi TEBİP 1. Kısa Sınavı

1. $\vec{a}$ ve $\vec{b}$ sabit vektörler ve $\vec{r} = x_i \hat{e_i}$ olmak üzere, aşağıdaki eşitliği *Levi-Civita* sembolünü kullanarak gösteriniz.

$$\vec{\nabla}\left(\vec{a}.\left(\vec{b}\times\vec{r}\right)\right) = \vec{a}\times\vec{b}$$

2. $\vec{a}$ ve $\vec{b}$ sabit olmayan vektörler olmak üzere, aşağıdaki eşitliğin sağlandığını *Levi-Civita* sembolünü kullanarak gösteriniz.

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{a}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{a}\right) - \nabla^2\vec{a}$$

i. $\vec{a}$'nın selenoidal bir vektör alanı olması durumunda,

ii. $\vec{a}$'nın *irrotasyone*l bir vektör alanı olması durumunda eşitliğin nasıl olacağını yazınız.

3. $r = \sqrt{(x_i)^2}$ olmak üzere, aşağıdaki ifadeleri hesaplayınız.

i. $\nabla^2 \ln r$ =

ii. $\vec{\nabla}.\left[r\,\vec{\nabla}\left(\frac{1}{r^n}\right)\right]$ =

4. Aşağıda verilen $\psi(x,y,z)$ fonksiyonunun laplasyenini hesaplayınız.

$$\psi(x,y,z) = e^{-x^2}\sin y + z^2$$

Başarılar 😊
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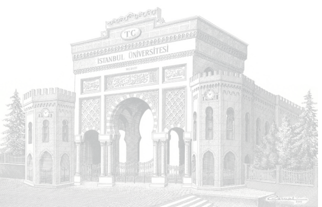
Not: Sınav süresi 30 dakikadır. Sadece 3 soru seçilmelidir.



1) $\vec{r} = x_i \hat{e}_i$

$\bar{a}, \bar{b}$: sabit vektörler

$$\bar{\nabla}\left(\bar{a}.(\bar{b}\times\bar{r})\right) = \partial_i \hat{e}_i \left(a_j \hat{e}_j . \left(\varepsilon_{mnl} b_m r_n \hat{e}_l\right)\right)$$

$$= \partial_i \hat{e}_i \left(a_j \hat{e}_j . \left[\varepsilon_{mnl} (b_m r_n) \hat{e}_l\right]\right)$$

$$= \partial_i \hat{e}_i \left[a_j \left(\varepsilon_{mnl} b_m r_n\right) \underbrace{\hat{e}_j . \hat{e}_l}_{\delta_{jl}}\right] \qquad \hat{e}_j . \hat{e}_l = \delta_{jl}$$

$j \to l$

$$= \partial_i \hat{e}_i \left[\varepsilon_{mnl} a_l b_m r_n\right]$$

$$= \varepsilon_{mnl} \underbrace{(\partial_i a_l)}_{0} b_m r_n \hat{e}_i + \varepsilon_{mnl} a_l \underbrace{(\partial_i b_m)}_{0} r_n \hat{e}_i + \varepsilon_{mnl} a_l b_m (\partial_i r_n) \hat{e}_i$$

a_l : sabit $\qquad$ b_m: sabit

$$= \varepsilon_{mnl} a_l b_m (\partial_i r_n) \hat{e}_i \qquad \partial_i r_n \neq 0 \Rightarrow \partial_i r_n = \delta_{in}$$

$$\partial_1 x_1 = 1, \quad \partial_2 x_2 = 1, \quad \partial_3 x_3 = 1$$

$$= \varepsilon_{mnl} a_l b_m \delta_{in} \hat{e}_i$$

$n \to i$

$$= \varepsilon_{mil} a_l b_m \hat{e}_i = \underbrace{\varepsilon_{lmi} a_l b_m \hat{e}_i}_{\bar{a}\times\bar{b}}$$

$$= \bar{a}\times\bar{b} \;//$$

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2) $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \varepsilon_{ijk}\, \partial_i \, (\vec{\nabla} \times \vec{a})_j \, \hat{e}_k$

$= \varepsilon_{ijk}\, \partial_i \, (\varepsilon_{mnj}\, \partial_m a_n)\, \hat{e}_k$

$= \varepsilon_{ijk}\, \varepsilon_{mnj}\, \partial_i (\partial_m a_n)\, \hat{e}_k \qquad \varepsilon_{ijk} = \varepsilon_{kij}$

$= \varepsilon_{kij}\, \varepsilon_{mnj}\, \partial_i (\partial_m a_n)\, \hat{e}_k$

$= \left[\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}\right] \partial_i (\partial_m a_n)\, \hat{e}_k$

$= \delta_{km}\delta_{in}\, \partial_i(\partial_m a_n)\, \hat{e}_k - \delta_{kn}\delta_{im}\, \partial_i (\partial_m a_n)\, \hat{e}_k$

$m \to k \quad n \to i \qquad\qquad k \to n \quad i \to m$

$= \partial_i (\partial_k a_i)\, \hat{e}_k - \partial_m (\partial_m a_n)\, \hat{e}_n$

$= \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{a}$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}$ //

i) $\vec{a}$ selenoidal vektör alanı ise; $\vec{\nabla} \cdot \vec{a} = 0$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\underbrace{\cancel{\vec{\nabla} \cdot \vec{a}}}_{0}) - \nabla^2 \vec{a} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = -\nabla^2 \vec{a}$ //

ii) $\vec{a}$ irrotasyonel (korunumlu) vektör alanı ise; $\vec{\nabla} \times \vec{a} = \vec{0}$

$\vec{\nabla} \times (\underbrace{\cancel{\vec{\nabla} \times \vec{a}}}_{0}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a} = 0 \Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) = \nabla^2 \vec{a}$ //

3) i) $\nabla^2 \ln r = \vec{\nabla}\cdot(\vec{\nabla} \ln r)$ $\quad\quad$ $\ln r = f(r) \Rightarrow \vec{\nabla} f(r) = \frac{\partial f(r)}{\partial r}\frac{dr}{dx_i}\hat{e}_i$

$$= \vec{\nabla}\cdot\left[\frac{\partial}{\partial r}(\ln r)\frac{dr}{dx_i}\hat{e}_i\right] \quad\quad \frac{dr}{dx_i}\hat{e}_i = 2x_i\frac{(x_i^2)^{-1/2}}{2}\hat{e}_i = \frac{\vec{r}}{r} = \hat{r}$$

$$= \vec{\nabla}\cdot\left[\frac{1}{r}\frac{\vec{r}}{r}\right]$$

$$= \vec{\nabla}\cdot\left[\frac{\vec{r}}{r^2}\right] = \vec{\nabla}\cdot\left[\vec{r}\,r^{-2}\right] = r^{-2}\underbrace{(\vec{\nabla}\cdot\vec{r})}_{3} + \vec{r}\cdot(\vec{\nabla} r^{-2})$$

$$= 3r^{-2} + \vec{r}\cdot\left(\frac{\partial}{\partial r}r^{-2}\left(\frac{dr}{dx_i}\hat{e}_i\right)\right)$$

$$= 3r^{-2} + \vec{r}\cdot\left(-2r^{-3}\left(\frac{\vec{r}}{r}\right)\right)$$

$$= 3r^{-2} - 2\left(r^{-3}\frac{\vec{r}\cdot\vec{r}}{r}\right) = 3r^{-2} - 2r^{-2} = r^{-2}$$

$$\nabla^2 \ln(r) = \frac{1}{r^2} \;//$$

ii) $\vec{\nabla}\cdot\left[r\,\vec{\nabla}\left(\frac{1}{r^n}\right)\right] = \vec{\nabla}\cdot\left[r\left(\frac{\partial}{\partial r}r^{-n}\right)\frac{dr}{dx_i}\hat{e}_i\right]$ $\quad\quad$ $\frac{dr}{dx_i}\hat{e}_i = \frac{\vec{r}}{r} = \hat{r}$

$$= \vec{\nabla}\cdot\left[r\left(-n\,r^{-n-1}\right)\frac{\vec{r}}{r}\right]$$

$$= \vec{\nabla}\cdot\left[-n\,r^{-n-1}\vec{r}\right] = -n\,\vec{\nabla}\cdot\left[\vec{r}\,r^{-n-1}\right]$$

$$= -n\left[\underbrace{(\vec{\nabla}\cdot\vec{r})}_{3}r^{-n-1} + \vec{r}\cdot\left(\vec{\nabla}r^{-n-1}\right)\right]$$

$$= -n\left[3r^{-n-1} + \vec{r}\cdot\left(\frac{\partial}{\partial r}r^{-n-1}\right)\underbrace{\frac{dr}{dx_i}\hat{e}_i}_{\vec{r}/r}\right]$$

$$= -n\left[3r^{-n-1} + \vec{r}\cdot\left[(-n-1)r^{-n-2}\frac{\vec{r}}{r}\right]\right]$$

$$= -n\left[3r^{-n-1} + (-n-1)r^{-n-1}\right] = -n\,r^{-n-1}(-n+2)$$

$$= r^{-(n+1)}(n^2-2n) \;//$$

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4) $\nabla^2 \psi(x,y,z) = \vec{\nabla}\cdot\left(\vec{\nabla}\psi(x,y,z)\right)$ $\qquad \psi(x,y,z) = e^{-x^2}\sin y + z^2$

$$\vec{\nabla} = \partial_i \hat{e}_i = \frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3$$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = (\partial_i\hat{e}_i)\cdot(\partial_j\hat{e}_j) = \partial_i\partial_j \underbrace{\hat{e}_i\cdot\hat{e}_j}_{\delta_{ij}} = \partial_i\partial_i = \partial_i^2$$

$$\vec{\nabla}\cdot\vec{\nabla} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Rightarrow \nabla^2\psi(x,y,z) = \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial z^2}\psi$$

- $\dfrac{\partial^2\psi}{\partial x^2} = \dfrac{\partial}{\partial x}\left[-2x e^{-x^2}\sin y\right] = -2\sin y \dfrac{\partial}{\partial x}\left(x e^{-x^2}\right)$

$$= -2\sin y\left(e^{-x^2} - 2x^2 e^{-x^2}\right)_{//}$$

- $\dfrac{\partial^2}{\partial y^2}\psi = \dfrac{\partial}{\partial y}\left[e^{-x^2}\cos y\right] = -e^{-x^2}\sin y_{//}$

- $\dfrac{\partial^2}{\partial z^2}\psi = \dfrac{\partial}{\partial z}\left[2z\right] = 2_{//}$

$$\Rightarrow \nabla^2\psi(x,y,z) = \underbrace{-2\sin y\, e^{-x^2} + 4\sin y\, x^2 e^{-x^2}}_{\frac{\partial^2\psi}{\partial x^2}} \underbrace{- e^{-x^2}\sin y}_{\frac{\partial^2\psi}{\partial y^2}} + \underbrace{2}_{\frac{\partial^2\psi}{\partial z^2}}$$

$$\nabla^2\psi(x,y,z) = -3\sin y\, e^{-x^2} + 4\sin y\, e^{-x^2}x^2 + 2$$

$$= \sin y\, e^{-x^2}\left(-3 + 4x^2\right) + 2_{//}$$

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