2.1. $Y_2^1(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$ küresel harmoniğini kullanarak $Y_2^2(\theta,\phi)$'yi oluşturunuz (L_+ operatörünü kullanın). Oluşturduğunuz $Y_2^2(\theta,\phi)$'nin,

a) $L_z=?$ b) $L=?$ c) $L^2=?$

$|Y_2^1(\theta,\phi)|^2$

$$Y_2^1(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}, \quad L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_+ Y_2^1(\theta,\phi) = Y_2^2(\theta,\phi)\,\underbrace{\hbar\sqrt{l(l+1)-m(m\pm1)}}_{\hbar\sqrt{2(2+1)-1(1+1)}=2\hbar} \Rightarrow L_+Y_2^1 = \hbar\sqrt{2}\,Y_2^2$$

$$\Rightarrow Y_2^2(\theta,\phi) = \frac{1}{\hbar\sqrt{2}}L_+Y_2^1$$

$$\Rightarrow Y_2^2(\theta,\phi) = \frac{1}{2\hbar}\left[\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\right]\left(-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}\right)$$

$$= \frac{-1}{2}\sqrt{\frac{15}{8\pi}}\,e^{i\phi}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\, e^{i\phi}\right) + i\cot\theta\frac{\partial}{\partial\phi}\left(\sin\theta\cos\theta\, e^{i\phi}\right)\right]$$

$$= \frac{-1}{2}\sqrt{\frac{15}{8\pi}}\,e^{i\phi}\left[e^{i\phi}\left(\cos^2\theta-\sin^2\theta\right) + i\cot\theta\, i\sin\theta\cos\theta\, e^{i\phi}\right]$$

$$= \frac{-1}{2}\sqrt{\frac{15}{8\pi}}\,e^{2i\phi}\left[\cancel{\cos^2\theta}-\sin^2\theta-\cancel{\cos^2\theta}\right] = +\sqrt{\frac{15}{4\cdot8\pi}}\sin^2\theta\, e^{2i\phi}$$

$|Y_2^2(\theta,\phi)|^2$

$$Y_2^2(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi} \;//$$

a: $L_z = -i\hbar\frac{\partial}{\partial\phi}$ $\quad L_z Y_2^2 = \lambda_{L_z} Y_2^2$

$$\Rightarrow -i\hbar\frac{\partial}{\partial\phi}\left(\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}\right) = \lambda_{L_z}\left(\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}\right)$$

$$\Rightarrow \lambda_{L_z} = 2\hbar \;// \qquad L_z Y_{nlm} = m\hbar\, Y_{nlm} \Rightarrow L_z Y_2^2 = 2\hbar Y_2^2 \;//$$

b:

$$L\,\psi_{n\ell m} = \hbar\sqrt{\ell(\ell+1)}\,\psi_{n\ell m} \qquad \psi_{n\ell m} = R_{n\ell}\,Y_\ell^m$$

$$L\,Y_2^2 = \hbar\sqrt{\ell(\ell+1)}\,Y_2^2$$

$$L\,\psi_2^2 = \hbar\sqrt{2(2+1)}\,Y_2^2 \longrightarrow L\,\psi_2^2 = \hbar\sqrt{6}\,\psi_2^2 \qquad \lambda_L = \sqrt{6}\,\hbar \,//$$

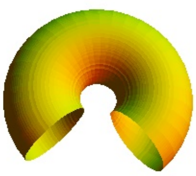
c:

$$L^2\,Y_2^2 = \hbar^2\,\ell(\ell+1)\,Y_2^2 \longrightarrow L^2\,Y_2^2 = \hbar^2\,2(2+1)\,Y_2^2$$

$$\lambda_{L^2} = 6\hbar^2 \,//$$

2.2. Bir Hidrojen atomu $n=4,\ \ell=3,\ m=3$ durumunda iken $L_x^2+L_y^2$ gözlenebilirini bir şekilde ölçebildiğimizi düşünelim. Hangi değerler hangi olasılıklarla elde edilir?

$|\psi_{4,3,3}|^2$



$$\left.\begin{array}{l} n=4 \\ \ell=3 \\ m=3 \end{array}\right\} \psi_{n\ell m} = \psi_{433}$$

$|n\ell m\rangle = |4,3,3\rangle$ (L ve L_z'nin özfonksiyonları)

$(L_x^2+L_y^2)\,|n\ell m\rangle = \lambda\,|n\ell m\rangle$ olsun

$L^2 = L_x^2+L_y^2+L_z^2 \Rightarrow L_x^2+L_y^2 = L^2 - L_z^2$ (1. yol)

$L_x = \dfrac{L_+ + L_-}{2},\quad L_y = \dfrac{L_+ - L_-}{2i}$ (2. yol)

• 1. yol;

$$(L_x^2+L_y^2)\,|4,3,3\rangle = \left[L^2 + (-L_z^2)\right]|4,3,3\rangle = \overbrace{L^2\,|4,3,3\rangle}^{\hbar^2\,3(3+1)} - \overbrace{L_z^2\,|4,3,3\rangle}^{(3\hbar)^2}$$

$$= \hbar^2\,3(3+1) - (3\hbar)^2 = 12\hbar^2 - 9\hbar^2$$

$$= 3\hbar^2 \,//$$ %100 olasılıkla. Çünkü ψ_{433} kararlı durumdadır.

• 2. yol;

$$L_x^2 = \left(\frac{L_+ + L_-}{2}\right)\left(\frac{L_+ + L_-}{2}\right) = \frac{1}{4}\left[L_+^2 + L_-L_+ + L_+L_- + L_-^2\right]$$

$$L_y^2 = \left(\frac{L_+ - L_-}{2i}\right)\left(\frac{L_+ - L_-}{2i}\right) = \frac{-1}{4}\left[L_+^2 - L_+L_- - L_-L_+ + L_-^2\right]$$

$$L_x^2 + L_y^2 = \frac{1}{4}\left[2L_+L_- + 2L_-L_+\right] = \frac{1}{2}\left[L_+L_- + L_-L_+\right]$$

$$(L_x^2 + L_y^2)|4,3,3\rangle = \frac{1}{2}\left[L_+L_- + L_-L_+\right]|4,3,3\rangle$$

$$= \frac{1}{2}L_+L_-|4,3,3\rangle + \frac{1}{2}L_-L_+|4,3,3\rangle$$

- $L_+L_-|4,3,3\rangle = L_+\left[\hbar\sqrt{3.4 - 3.(3-1)}\;|4,3,2\rangle\right]$

$$= L_+\,\hbar\sqrt{6}\,|4,3,2\rangle$$

$$= \hbar\sqrt{6}\;L_+|4,3,2\rangle$$

$$= \hbar\sqrt{6}\left[\hbar\sqrt{3.4 - 2(2+1)}\right] = \hbar\sqrt{6}\left(\hbar\sqrt{6}\right)|4,3,3\rangle$$

$$= 6\hbar^2|4,3,3\rangle$$

- $L_-L_+|4,3,3\rangle = L_-\left[\underbrace{L_+|4,3,3\rangle}_{0}\right] = L_-\left[\hbar\underbrace{\sqrt{\cancel{3.4 - 3.4}}}_{0}\;\cancel{|4,3,4\rangle}\right] = 0$

$$\Rightarrow \left[L_x^2 + L_y^2\right]|4,3,3\rangle = \frac{1}{2}\left[6\hbar^2|4,3,3\rangle + 0|4,3,3\rangle\right]$$

$$= 3\hbar^2 \;//$$

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2.3. Bir Hidrojen atomunun başlangıç dalga fonksiyonu $n=2,\ l=1,\ m=1$ ve $n=2,\ l=1,\ m=-1$ kararlı durumların lineer kombinasyonudur. Yani;

$$\Psi = A\left(\psi_{211} + \psi_{21-1}\right)$$

dir.

a) Dalga fonksiyonunu normalleştiriniz.

b) Enerji ölçüldüğünde hangi değerler hangi olasılıklarla elde edilir?

c) $\langle E \rangle = ?$

d) Yörüngesel açısal momentum ölçüldüğünde hangi değerler hangi olasılıklar ile elde edilir?

e) $\langle L \rangle = ?$

f) Yörüngesel açısal momentumun z bileşeni ölçüldüğünde hangi değerler hangi olasılıklar ile elde edilir?

g) $\langle L_z \rangle = ?$

a: $\langle \Psi | \Psi \rangle = 1$ olmalı; $|\psi_{211}\rangle = |2,1,1\rangle$, $|\psi_{21-1}\rangle = |2,1,-1\rangle$

$$1 = \langle \Psi | \Psi \rangle = A^{*}\left[\langle 211| + \langle 2,1-1|\right]\left[|211\rangle + |21-1\rangle\right] \qquad \langle n'l'm'|nlm\rangle = \delta_{nn'}\delta_{ll'}\delta_{mm'}$$

$$\frac{1}{|A|^2} = \Big[\underbrace{\langle 211|211\rangle}_{\delta_{22}\delta_{11}\delta_{11}=1} + \underbrace{\langle 211|21-1\rangle}_{\delta_{22}\delta_{11}\delta_{1-1}=0} + \underbrace{\langle 21-1|211\rangle}_{\delta_{22}\delta_{11}\delta_{-11}=0} + \underbrace{\langle 21-1|21-1\rangle}_{\delta_{22}\delta_{11}\delta_{-1-1}=1}\Big]$$

$$\frac{1}{A^2} = (1+1) \Rightarrow A^{-1} = \sqrt{2} \Rightarrow A = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \Psi = \frac{1}{\sqrt{2}}\left(\psi_{211} + \psi_{21-1}\right)$$

b: $H|n,l,m\rangle = E_n|n,l,m\rangle, \quad E_n = -\frac{13.6eV}{n^2}$

$$\Psi = \frac{1}{\sqrt{2}}\left[\psi_{211} + \psi_{21-1}\right]$$

• $H|\Psi\rangle = H\left[\frac{1}{\sqrt{2}}|\psi_{211}\rangle + |\psi_{21-1}\rangle\right] = H\left[|211\rangle + |21-1\rangle\right]\frac{1}{\sqrt{2}}$

$$= \left(H|211\rangle + H|21-1\rangle\right)\frac{1}{\sqrt{2}}$$

$$= \left(E_2|211\rangle + E_2|21-1\rangle\right)\frac{1}{\sqrt{2}}$$

$= E_2|\psi\rangle$ $\quad$ $|n,l,m\rangle$ durumları E_n'nin özfonksiyonlarıdır, %100 olasılıkla $E_2 = -\frac{13.6eV}{\;}$ bulunur

c: $\langle E\rangle = \langle\psi|H|\psi\rangle$

$$= \frac{1}{2}\left[\langle 211| + \langle 21-1|\right] H \left[|211\rangle + |21-1\rangle\right]$$

$$= \frac{1}{2}\left[\langle 211| + \langle 21-1|\right]\left(E_2|211\rangle + E_2|21-1\rangle\right)$$

$$= \frac{1}{2}\left(E_2\langle 211|211\rangle + E_2\langle 21-1|21-1\rangle\right)$$

$$= \frac{2E_2}{E_2} \Rightarrow \langle E\rangle = E_2 \,//$$

d: $L|nlm\rangle = \hbar\sqrt{l(l+1)}\,|nlm\rangle$

$$L|\psi\rangle = L\left[\frac{1}{\sqrt{2}}\left(|211\rangle + |21-1\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\underbrace{L|211\rangle}_{\hbar\sqrt{(1+1)1}\,|211\rangle} + L|21-1\rangle\right] = \frac{1}{\sqrt{2}}\left[\hbar\sqrt{1(1+1)}\,|211\rangle + \hbar\sqrt{1(1+1)}\,|21-1\rangle\right]$$

$$= 2\hbar\sqrt{2}\left[\frac{1}{\sqrt{2}}\left(|211\rangle + |21-1\rangle\right)\right] = \sqrt{2}\,\hbar|\psi\rangle$$

Kuantum Fiziği 2 Uygulama Notları
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$L|\Psi\rangle = 2\sqrt{2}\hbar|\Psi\rangle$ $|21m\rangle$ L 'nin özfonksiyonlarıdır. %100 olasılıkla $2\sqrt{2}\hbar$ bulunur.

e:

$$\langle L\rangle = \langle\Psi|L|\Psi\rangle$$

$$= \frac{1}{2}\left[\langle 211| + \langle 21\text{-}1|\right] L \left[|211\rangle + |21\text{-}1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\langle 211| + \langle 21\text{-}1|\right]\left(L|211\rangle + L|21\text{-}1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[\langle 211| + \langle 21\text{-}1|\right]\left(\hbar\sqrt{1(1+1)}\,|211\rangle + \hbar\sqrt{1(1+1)}\,|21\text{-}1\rangle\right)$$

$$= 2\sqrt{2}\hbar\,\langle\Psi|\Psi\rangle$$

$$\langle L\rangle = 2\sqrt{2}\hbar \,//$$

f: $L_z|n\ell m\rangle = m\hbar|n\ell m\rangle$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|211\rangle + |21\text{-}1\rangle\right]$$

$$P(L_z=\hbar) = \left|\langle 211|\Psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\underbrace{\langle 211|211\rangle}_{1} + \cancel{\langle 211|21\text{-}1\rangle}^{0}\right)\right|^2$$

$$P(L_z=\hbar) = \frac{1}{2}\,// \qquad P(L_z=-\hbar) = +\,1/2\,//$$

g: $$\langle L_z\rangle = \sum_i P_i L_{z_i} = P(L_z=\hbar)\,\hbar + P(L_z=-\hbar)(-\hbar)$$

$$= \frac{1}{2}\hbar + \frac{1}{2}(-\hbar) = 0$$

$$\langle L_z\rangle = 0\,//$$