1) $t=0$ anında bir parçacık;

$$\Psi(x,t=0)=\begin{cases} A\dfrac{x}{a}, & 0\leq x\leq a, \\ A\dfrac{(b-x)}{(b-a)}, & a\leq x\leq b \\ 0, & \text{diğer yerler} \end{cases}$$

ile verilen dalga fonksiyonu ile temsil edilmektedir. Burada A, a ve b birer sabittir.

a) Ψ'yi normalleyiniz.

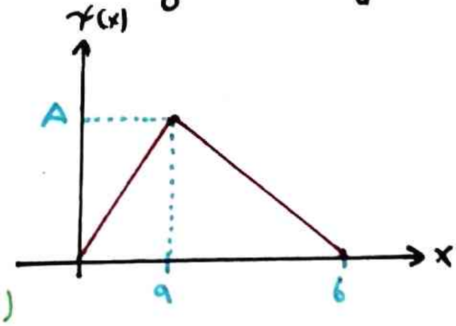
b) $\Psi(x,0)$'ı x'in fonksiyonu olarak çiziniz.

c) $t=0$ anında parçacık en olası nerede bulunur?

d) Parçacığın a'nın solunda bulunma olasılığı nedir?

a:
$$1=\langle\Psi|\Psi\rangle=\int_{-\infty}^{+\infty}\Psi^*\Psi\,dx=\int_{-\infty}^{0}\underbrace{|\Psi|^2}_{0}dx+\int_0^a|\Psi|^2dx+\int_a^b|\Psi|^2dx+\int_b^{\infty}\underbrace{|\Psi|^2}_{0}dx$$

$$\Rightarrow 1=\int_0^a\underbrace{|\Psi|^2}_{\Psi^*(0<x<a)\,\Psi(0<x<a)}dx+\int_a^b\underbrace{|\Psi|^2}_{\Psi^*(a<x<b)\,\Psi(a<x<b)}dx$$

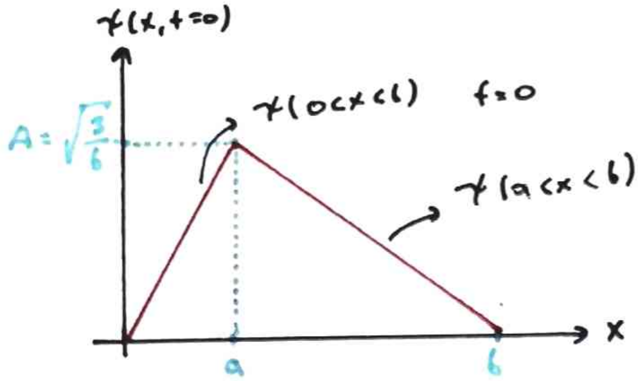


$$\Rightarrow 1=A^*A\int_0^a\left(\frac{x}{a}\right)^*\left(\frac{x}{a}\right)dx+A^*A\int_a^b\left[\frac{(b-x)}{(b-a)}\right]^*\left[\frac{(b-x)}{(b-a)}\right]dx$$

$$1=|A|^2\left[\int_0^a\frac{x^2}{a^2}dx+\int_a^b\frac{(b-x)^2}{(b-a)^2}dx\right]\Rightarrow\frac{1}{|A|^2}=\frac{1}{a^2}\int_0^a x^2dx+\frac{1}{(b-a)^2}\int_a^b(b^2-2bx+x^2)dx$$

$$\Rightarrow\frac{1}{|A|^2}=\frac{1}{a^2}\left[\frac{x^3}{3}\right]_{x=0}^{x=a}+\frac{1}{(b-a)^2}\left[b^2x-\frac{2bx^2}{2}+\frac{x^3}{3}\right]_{x=a}^{x=b}\Longrightarrow A=\sqrt{\frac{3}{b}}\;//$$

b:



c: $\psi(x,t=0)$'ın maksimum değeri aldığı noktada olasılık yoğunluğu $|\psi(x,t=0)|^2$'da maksimum değeri alır.

$x=a$ noktası!!

d: $$P(-\infty<x<a) = \int_{-\infty}^{a} |\psi(x,t=0)|^2 dx = \int_{-\infty}^{0} \underbrace{|\psi(x,t=0)|^2}_{0} dx + \int_{0}^{a} \underbrace{|\psi(x,t=0)|^2}_{\psi^*(0<x<a)\,\psi(0<x<a)} dx$$

$$P(-\infty<x<a) = \int_{0}^{a} |A|^2 \frac{x^2}{a^2} dx = \Big(\underbrace{\sqrt{\tfrac{3}{b}}}_{A}\Big)^2 \frac{1}{a^2}\left[\frac{x^3}{3}\right]_0^a = \frac{3}{b}\,\frac{1}{a^2}\left[\frac{a^3}{3} - 0\right]$$

$$P(-\infty<x<a) = \frac{a}{b} \;//$$

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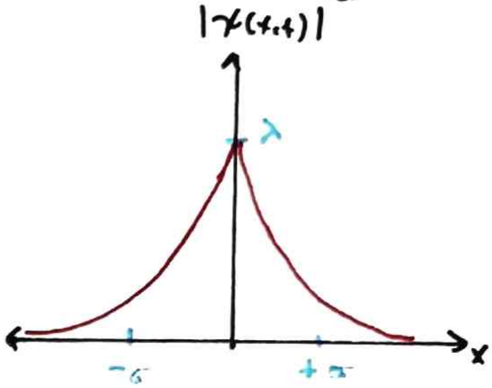
2) $\psi(x,t) = A\, e^{-\lambda|x|}\, e^{-i\omega t}$ ile verilen bir dalga fonksiyonu için;

(A, λ ve ω birer pozitif reel sabit)

a) ψ'yi normalleyiniz.

b) x ve x^2'nin beklenen değerlerini bulunuz.



a:

$$|\psi(x,t)\rangle = |\psi(x,t=0)\rangle\, e^{-iEt/\hbar}$$

$$|\psi(x,t)\rangle = \underbrace{A\, e^{-\lambda|x|}}_{|\psi(x,t=0)\rangle}\, e^{-i\omega t}, \quad E=\hbar\omega$$

$$1 = \langle\psi(x,t)|\psi(x,t)\rangle = \int_{-\infty}^{+\infty} |\psi(x,t)|^2 dx = |A|^2 \int_{-\infty}^{+\infty} e^{-2\lambda|x|}\, \underbrace{e^{+i\omega t}}_{(e^{-i\omega t})^*}\, e^{-i\omega t}\, dx$$

$$1 = |A|^2 \int_{-\infty}^{+\infty} e^{-2\lambda|x|}\, e^{0}\, dx \Rightarrow \frac{1}{|A|^2} = 2\int_{0}^{\infty} e^{-2\lambda x} dx = -\frac{2}{2\lambda}\Big[e^{-2\lambda x}\Big]_{x=0}^{x=\infty}$$

$$\Rightarrow |A|^2 = \lambda \Rightarrow A = \sqrt{\lambda}$$

b:

$$\bullet\ \langle x\rangle = \langle\psi(x,t)|x|\psi(x,t)\rangle = \int_{-\infty}^{+\infty} A^* e^{-\lambda|x|}\, e^{+i\omega t}\, x\, A\, e^{-\lambda|x|}\, e^{-i\omega t} dx$$

$$\langle x\rangle = \underset{|A|^2}{\lambda}\, e^{0} \int_{-\infty}^{+\infty} \underbrace{x\, e^{-2\lambda|x|}}_{\text{tek fonksiyon}} dx = \lambda\underbrace{\int_{-\infty}^{0} x\, e^{-2\lambda|x|} dx}_{\substack{x \to -x \\ dx \to -dx}} + \lambda\int_{0}^{\infty} x\, e^{-2\lambda|x|} dx$$

$$\langle x\rangle = \lambda\int_{+\infty}^{0} (-x)\, e^{-2\lambda|x|}(-dx) + \lambda\int_{0}^{\infty} x\, e^{-2\lambda x} dx = -\lambda\underbrace{\int_{0}^{\infty} x\, e^{-2\lambda|x|} dx}_{I} + \lambda\underbrace{\int_{0}^{\infty} x\, e^{-2\lambda|x|} dx}_{I}$$

$$I - I = 0$$

$$\langle x \rangle = 0 \;//$$

$$\bullet \; \langle x^2 \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t)\, x^2\, \psi(x,t)\, dx = |A|^2 \int_{-\infty}^{+\infty} e^{-2\lambda|x|}\, x^2\, e^{-i\omega t + i\omega t}\, dx$$

$$\langle x^2 \rangle = \underbrace{\lambda}_{|A|^2} e^0 \int_{-\infty}^{+\infty} x^2 e^{-2\lambda|x|}\, dx = 2\lambda \int_{0}^{\infty} x^2 e^{-2\lambda x}\, dx$$

$n = 2$

$a = 1/2\lambda$

$$I = \int_0^{\infty} u^n e^{-u/a}\, du = n!\, a^{n+1}$$

$$\langle x^2 \rangle = 2\lambda \left[2! \left(\frac{1}{2\lambda} \right)^{2+1} \right] \Rightarrow \langle x^2 \rangle = \frac{1}{2\lambda^2} \;//$$

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3) m kütleli bir parçacık,

$$\psi(x,t) = A\, e^{-a\left[(mx^2/\hbar)+it\right]}$$

ile verilen bir durumdadır. Burada A ve a pozitif ve reel sayılardır.

a) ψ'yi normalleyiniz.

b) ψ hangi $V(x)$ potansiyeli için Schrödinger denklemini sağlar?

c) x, x^2, p ve p^2'nin beklenen değerlerini hesaplayınız.

$$|\psi(x,t)\rangle = |\psi(x,t=0)\rangle\, e^{-iEt/\hbar}$$

$$|\psi(x,t)\rangle = A\, e^{-a\left[(mx^2/\hbar)+it\right]} = \underbrace{A\, e^{-a\frac{mx^2}{\hbar}}}_{|\psi(x,t=0)\rangle}\ \underbrace{e^{-iat}}_{e^{-i\omega t}} \qquad a = \frac{E}{\hbar} = \omega$$

a: $$1 = \langle\psi(x,t)|\psi(x,t)\rangle = \int_{-\infty}^{+\infty} |A|^2 e^{-2\frac{amx^2}{\hbar}} e^{+iat} e^{-iat}\, dx = 2|A|^2 \int_0^{\infty} e^{-\frac{2amx^2}{\hbar}} dx$$

$$\frac{1}{|A|^2} = 2\int_0^{\infty} e^{-\frac{2am}{\hbar}x^2} dx \qquad n=0,\ \alpha = \sqrt{\frac{\hbar}{2am}}$$

$$I = \int_0^{\infty} x^{2n} e^{-x^2/\alpha^2} dx = \sqrt{\pi}\, \frac{(2n)!}{n!} \left(\frac{\alpha}{2}\right)^{2n+1}$$

$$\Rightarrow \frac{1}{|A|^2} = 2\left[\sqrt{\pi}\, \frac{(2\cdot 0)!}{0!} \left(\frac{1}{2}\sqrt{\frac{\hbar}{2am}}\right)^{2\cdot 0+1}\right] = \sqrt{\frac{\hbar\pi}{2am}} \Rightarrow A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}$$

b: $$H|\psi(x,t)\rangle = E|\psi(x,t)\rangle \qquad p = -i\hbar\vec{\nabla} \Rightarrow p_x = -i\hbar\frac{d}{dx}$$

$$p_x^2 = -\hbar^2\frac{d^2}{dx^2}$$

$$\underbrace{\left[\frac{p_x^2}{2m} + V(x)\right]}_{H} |\psi(x,t)\rangle = E\,|\psi(x,t)\rangle \qquad E = i\hbar\frac{d}{dt}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}|\psi(x,t)\rangle + V(x)\,|\psi(x,t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(x,t)\rangle$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi = +i\hbar\frac{\partial \psi}{\partial t}$$

- $$\frac{\partial^2}{\partial x^2}\left[A e^{-\frac{am}{\hbar}x^2} e^{-iat}\right] = \frac{\partial}{\partial x}\left[\left(\frac{-2amx}{\hbar}\right) A e^{-\frac{am}{\hbar}x^2} e^{-iat}\right]$$

$$\frac{\partial^2}{\partial x^2}\psi = A\left(\frac{-2am}{\hbar}\right)e^{-iat}\frac{\partial}{\partial x}\left[x e^{-\frac{am}{\hbar}x^2}\right]$$

$$\frac{\partial^2}{\partial x^2}\psi = A\left(\frac{-2am}{\hbar}\right)e^{-iat}\left[1 e^{-\frac{am}{\hbar}x^2} + x\left(\frac{-2amx}{\hbar}\right)e^{-\frac{am}{\hbar}x^2}\right]$$

$$\frac{\partial^2}{\partial x^2}\psi = -\frac{2am}{\hbar}\underline{\psi(x,t)} + \left(\frac{2am}{\hbar}\right)^2 x^2 \underline{\psi(x,t)}$$

- $$\frac{\partial \psi(x,t)}{\partial t} = \frac{\partial}{\partial t}\left[A e^{-\frac{am}{\hbar}x^2} e^{-iat}\right] = -ia\,\psi(x,t)$$

$$\Rightarrow \frac{-\hbar^2}{2m}\left[-\frac{2am}{\hbar}\psi + \left(\frac{2am}{\hbar}\right)^2 x^2\psi\right] + V(x)\psi = +i\hbar(-ia)\psi$$

$$\Rightarrow V(x)\psi = \hbar a\psi - \hbar a\psi + \frac{2a^2 m x^2 \cancel{\hbar^2}}{\cancel{\hbar^2}}\psi$$

$$V(x) = \hbar a - \hbar a + 2a^2 m x^2 \Rightarrow V(x) = 2ma^2x^2 //$$

c:

$$\langle x\rangle = \langle\psi|x|\psi\rangle = \int_{-\infty}^{+\infty} |A|^2 e^{-\frac{2am}{\hbar}x^2} x\, e^{-iat+iat}\,dx$$

$$\langle x\rangle = |A|^2\int_{-\infty}^{+\infty} x e^{-\frac{2am}{\hbar}x^2}dx \qquad I = \int_0^{\infty} x^{2n} e^{-x^2/\alpha^2}dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{\alpha}{2}\right)^{2n+1}$$

$$n = 1/2 \quad \alpha = \left(\frac{\hbar}{2am}\right)^{1/2} \qquad n = 1/2 \Rightarrow I = 0$$

$$\langle x\rangle = |A|^2\, 2\int_0^{\infty} \underbrace{x e^{-(2am/\hbar)x^2}}_{\text{tek fon.}}dx \Rightarrow \langle x\rangle = 0 //$$

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$$\bullet\ \langle x^2\rangle = \langle \psi | x^2 | \psi\rangle = |A|^2 \int_{-\infty}^{+\infty} e^{-\frac{2am}{\hbar}x^2}\, x^2\, e^{-iat+iat}\, dx$$

$$\langle x^2\rangle = |A|^2\, 2\int_{0}^{+\infty} x^2 e^{-\frac{2am}{\hbar}x^2} dx \qquad \int_0^{\infty} x^{2n} e^{-x^2/\alpha^2} dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{\alpha}{2}\right)^{2n+1}$$

$n=1,\ \alpha = \left(\frac{\hbar}{2am}\right)^{1/2}$

$$\langle x^2\rangle = |A|^2\, 2\left[\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{1}{2}\sqrt{\frac{\hbar}{2am}}\right)^{2\cdot 1+1}\right] \Rightarrow \langle x^2\rangle = \frac{\hbar}{4am}\ //$$

$$\bullet\ \langle p\rangle = \langle \psi | p | \psi\rangle = \langle \psi | \left(-i\hbar\frac{d}{dx}\right) | \psi\rangle$$

$$\langle p\rangle = |A|^2 \int_{-\infty}^{+\infty} e^{-\frac{am}{\hbar}x^2}\underbrace{\left(-i\hbar\frac{d}{dx}\right)\left(e^{-\frac{am}{\hbar}x^2}\right)}_{p|\psi\rangle}\underbrace{e^{-iat+iat}}_{e^0=1}\, dx$$

$$\langle p\rangle = -i\hbar|A|^2 \int_{-\infty}^{+\infty} e^{-\frac{am}{\hbar}x^2}\left[\left(-\frac{am2}{\hbar}\right)x\, e^{-\frac{am}{\hbar}x^2}\right]dx$$

$$\langle p\rangle = +i\hbar\left(\frac{2am}{\hbar}\right)|A|^2 \int_{-\infty}^{+\infty} \underbrace{x\, e^{-\frac{2am}{\hbar}x^2}}_{\substack{\text{tek fonk}\\ I=0}} dx = 0 \Rightarrow \langle p\rangle = 0\ //$$

İkinci yolu

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} \Rightarrow \langle x\rangle = 0,\ \langle p\rangle = 0\ //$$

$\bullet\ \langle p^2\rangle = am\hbar\ //$ (Ödev!!!!)

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