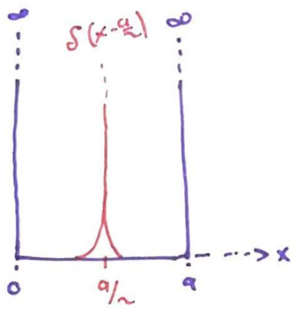
7.1. Sonsuz kare kuyu potansiyelin merkezinde bir delta fonksiyon çıkıntısını koyduğumuzu varsayalım:

$$H' = \alpha\,\delta\left(x - \frac{a}{2}\right)$$

ve burada α bir sabittir.

a) İzin verilen enerji değerlerine birinci dereceden düzeltmesi hesaplayın. n'in çift değerleri için enerjinin neden tedirgenmediğini açıklayınız.

b) ψ_1^1 taban durum düzeltme serisinde sıfırdan farklı ilk üç terimi hesaplayınız.



$$H = \underbrace{-\frac{\hbar^2}{2m}\nabla^2 + V}_{H^0} + \underbrace{\delta\left(x-\frac{a}{2}\right)\alpha}_{H'} \qquad H' = \alpha\,\delta\left(x-\frac{a}{2}\right)$$

$$H^0\psi_n^0 = E_n^0\psi_n^0 \implies \psi_n^0 = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$E_n^0 = \frac{\hbar^2\pi^2}{2ma^2}n^2$$

$$E_n^1 = \frac{\langle\psi_n^0|H'|\psi_n^0\rangle}{\underbrace{\langle\psi_n^0|\psi_n^0\rangle}_{\delta_{nn}=1}} = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\psi_m^0$$

a: $E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle$, $n=1 \Rightarrow E_1^1 = \langle\psi_1^0|H'|\psi_1^0\rangle$, $\psi_1^0 = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)$

$$E_1^1 = \int_0^a \psi_1^{0*}\underbrace{\alpha\,\delta\left(x-\frac{a}{2}\right)}_{H'}\psi_1^0\,dx = \frac{2\alpha}{a}\underbrace{\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\delta\left(x-\frac{a}{2}\right)dx}$$

$$\int f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

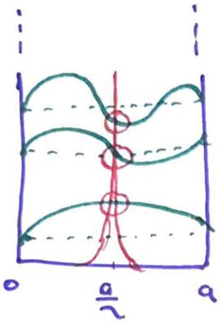
$$E_1^1 = \frac{2\alpha}{a}\sin^2\left(\frac{\pi}{a}\frac{a}{2}\right) = \frac{2\alpha}{a}\sin^2\frac{\pi}{2}$$

$E_1^1 = \frac{2\alpha}{a}$ ($n=1$ için enerjiye gelen 1. dereceden düzeltme)

Çözümü tüm n durumlara genelleyelim;

$$E_n^0 = \langle \psi_n^0 | H' | \psi_n^0 \rangle = \int_0^a \psi_n^{0*}\, \alpha\, \delta\left(x-\frac{a}{2}\right) \psi_n^0 \, dx = \frac{2\alpha}{a} \underbrace{\int_0^a \sin^2\left(\frac{n\pi}{a}x\right) \delta\left(x-\frac{a}{2}\right) dx}_{\sin^2\left(\frac{n\pi}{a}\frac{a}{2}\right)}$$

$$E_n^0 = \frac{2\alpha}{a}\sin^2\left(\frac{n\pi}{2}\right) \begin{cases} 0, & n \text{ çift ise} \\ \frac{2\alpha}{a}, & n \text{ tek ise} \end{cases}$$



n'in çift olduğu durumlarda, pertürbasyon teriminin bulunduğu $\frac{a}{2}$ noktasında dalga fonksiyonu sıfırdır. Yani bu durumlar $H' = \alpha\delta(x-\frac{a}{2})$ pertürbasyonunu "hissetmezler".

b:
$$|\psi_n^1\rangle = \sum_{m\neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{(E_n^0 - E_m^0)} |\psi_m^0\rangle$$

$$n=1 \Rightarrow |\psi_1^1\rangle = \sum_{m\neq 1} \frac{\langle \psi_m^0 | H' | \psi_1^0 \rangle}{(-E_m^0 + E_1^0)} |\psi_m^0\rangle$$

$$\bullet\ \langle \psi_m^0 | H' | \psi_1^0 \rangle = \int_0^a \psi_m^{0*}\, \alpha\, \delta\left(x-\frac{a}{2}\right)\psi_1^0\, dx = \frac{2\alpha}{a}\underbrace{\int_0^a \sin\left(\frac{m\pi}{a}x\right)\delta\left(x-\frac{a}{2}\right)\sin\left(\frac{\pi}{a}x\right)dx}_{\sin\left(\frac{m\pi}{a}\frac{a}{2}\right)\sin\left(\frac{\pi}{a}\frac{a}{2}\right)}$$

$$= \frac{2\alpha}{a}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{m\pi}{2}\right)\frac{2\alpha}{a}$$

$= \frac{2\alpha}{a}\sin\left(\frac{m\pi}{2}\right)$, m'in tek değerleri için sıfırdan farklı,

$m = 3, 5, 7, 9 \ldots \Rightarrow \neq 0$

$$\bullet\ -E_m^0 + E_1^0 = -\frac{\hbar^2\pi^2}{2ma^2}m^2 + \frac{\hbar^2\pi^2}{2ma^2}1^2 = \frac{\hbar^2\pi^2}{2ma^2}(1-m^2)$$

$$\Rightarrow |\psi_1^1\rangle = \sum_{m=1,3,5} \frac{\frac{2\alpha}{a}\sin\frac{m\pi}{a}}{\frac{\hbar^2\pi^2}{2ma^2}(1-m^2)} |\psi_m^0\rangle$$

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$$\psi_1^1 = \frac{4\alpha m a}{\hbar \pi^2}\left[\frac{\sin\left(\frac{3\pi}{2}\right)}{(1-3^2)}\psi_3^0 + \frac{\sin\left(\frac{5\pi}{2}\right)}{(1-5^2)}\psi_5^0 + \frac{\sin\left(\frac{7\pi}{2}\right)}{(1-7^2)}\psi_7^0\right]$$

$$\psi_1^1 = \frac{4\alpha m a}{\hbar \pi^2}\left[\frac{1}{8}\psi_3^0 - \frac{1}{24}\psi_5^0 + \frac{1}{48}\psi_7^0\right] //$$

7.2. Harmonik salınıcı potansiyeli için yay sabitini söz an arttırdığımızda $k \rightarrow k' = (1+\epsilon)k$ olduğunu varsayın. (Belki bunu yayı soğutarak yapabiliriz).

Enerjiye gelen 1. dereceden düzeltmesi hesaplayınız.

$$H^0 = \underbrace{\frac{-\hbar^2}{2m}\nabla^2}_{T} + \underbrace{\frac{1}{2}kx^2}_{U(x)}, \quad E_n^0 = \left(n+\frac{1}{2}\right)\hbar\omega,$$

$$k \rightarrow k' = (1+\epsilon)k$$

$$V(x) = \frac{1}{2}kx^2 \rightarrow \frac{1}{2}kx^2(1+\epsilon)$$

$$V(x)' = \frac{1}{2}kx^2 + \epsilon\frac{1}{2}kx^2$$

$$\epsilon \ll 1$$

$$H = \underbrace{\frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}kx^2}_{H^0} + \underbrace{\frac{\epsilon}{2}kx^2}_{H'}$$

$$\Rightarrow E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle = \langle n | \frac{\epsilon}{2}kx^2 | n \rangle$$

$$x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a_+ + a_-)$$

$$a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a_-|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$x^2 = \left(\frac{\hbar}{2m\omega}\right)\left[a_+^2 + a_+a_- + a_-a_+ + a_-^2\right]$$

$$a_+a_- \neq a_-a_+$$

$$\Rightarrow E_n^1 = \frac{\epsilon}{2}k\left(\frac{\hbar}{2m\omega}\right)\left[\langle n|\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right)|n\rangle\right]$$

$$E_n^1 = \frac{\epsilon k\hbar}{4m\omega}\left[\underbrace{\langle n|a_+^2|n\rangle}_{1} + \underbrace{\langle n|a_+a_-|n\rangle}_{2} + \underbrace{\langle n|a_-a_+|n\rangle}_{3} + \underbrace{\langle n|a_-^2|n\rangle}_{4}\right]$$

$$1 \rightarrow \langle n|a_+a_+|n\rangle = \langle n|a_+\sqrt{n+1}\,|n+1\rangle = \sqrt{n+1}\,\langle n|a_+|n+1\rangle$$

$$= \sqrt{n+1}\,\langle n|\sqrt{n+2}\,|n+2\rangle = \sqrt{n+1}\sqrt{n+2}\,\cancel{\langle n|n+2\rangle}^{\,0}$$

$$= 0 //$$

② → $\langle n|a_+a_-|n\rangle = \langle n|a_+\sqrt{n}\,|n-1\rangle = \sqrt{n}\,\langle n|a_+|n-1\rangle$

$= \sqrt{n}\,\langle n|\sqrt{n}\,|n\rangle = n\,\underbrace{\langle n|n\rangle}_{1}$

$= n$ //

③ → $\langle n|a_-a_+|n\rangle = \langle n|a_-\sqrt{n+1}\,|n+1\rangle = \sqrt{n+1}\,\langle n|a_-|n+1\rangle$

$= \sqrt{n+1}\,\langle n|\sqrt{n+1}\,|n\rangle = n+1\,\underbrace{\langle n|n\rangle}_{1}$

$= n+1$ //

④ → $\langle n|a_-^2|n\rangle = \langle n|a_-a_-|n\rangle = \langle n|a_-\sqrt{n}\,|n-1\rangle$

$= \sqrt{n}\,\langle n|a_-|n-1\rangle = \sqrt{n}\,\langle n|\sqrt{n-1}\,|n-2\rangle$

$= \sqrt{n}\sqrt{n-1}\,\cancelto{0}{\langle n|n-2\rangle}$

$= 0$ //

$\Rightarrow E_n^1 = \frac{\varepsilon k \hbar}{4m\omega}\left[0 + n + n+1 + 0\right] = \frac{\varepsilon k \hbar}{4m\omega}(2n+1) = \frac{\varepsilon k \hbar}{2m\omega}(n+1/2)$

$\omega = \sqrt{\frac{k}{m}} \Rightarrow E_n^1 = \frac{1}{2}\varepsilon\hbar\omega(n+1/2) = \varepsilon\frac{E_n^0}{2}$ //

$\langle V\rangle = \langle T\rangle \Rightarrow \langle V\rangle = \frac{E_n^0}{2},\ \langle T\rangle = \frac{E_n^0}{2}$

$\langle V\rangle + \langle T\rangle = E_n^0$

$\Rightarrow E_n^1 = \frac{\varepsilon}{2}E_n^0$ //

7.3. Bir boyutlu harmonik salınıcı potansiyeli içinde bulunan yüklü bir parçacığı göz önüne alalım. Potansiyel enerjisi $H' = -qEx$ artıracak kadar da zayıf bir elektrik alan uyguladığımızı düşünelim.

Enerji düzeylerinde 1. derecede değişimin olmadığını gösteriniz.

Enerjiye gelen 2. derecede düzeltme terimini hesaplayınız.

$$E_n^0 = \left(n+\frac{1}{2}\right)\hbar\omega$$

$$H = \underbrace{-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}kx^2}_{H^0} \underbrace{- qEx}_{H'}$$

$$H^0|n\rangle = E_n^0|n\rangle$$

- $E_n^1 = \langle \psi_n^0 | H' | \psi_n^0\rangle = \langle n|(-qEx)|n\rangle = -qE\langle n|x|n\rangle$

$$x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a_+ + a_-)$$

$$\Rightarrow E_n^1 = -qE\langle n|\left[\left(\frac{\hbar}{2m\omega}\right)^{1/2}(a_+ + a_-)\right]|n\rangle$$

$$E_n^1 = -qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\langle n|(a_+ + a_-)|n\rangle = -qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\underbrace{\langle n|a_+|n\rangle}_{\sqrt{n+1}\,\underbrace{\langle n|n+1\rangle}_{0}} + \underbrace{\langle n|a_-|n\rangle}_{\langle n|\sqrt{n}|n-1\rangle = \sqrt{n}\,\underbrace{\langle n|n-1\rangle}_{0}}\right]$$

$$E_n^1 = -qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\sqrt{n+1}\,\underbrace{\langle n|n+1\rangle}_{\delta_{n,n+1}=0} + \sqrt{n}\,\underbrace{\langle n|n-1\rangle}_{\delta_{n,n-1}=0}\right]$$

$$E_n^1 = 0$$

- $E_n^2 = \sum_{m\neq n}\dfrac{|\langle \psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0}$

$$\circ\ \langle \psi_m^0|H'|\psi_n^0\rangle = -qE\langle m|x|n\rangle = -qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\langle m|(a_+ + a_-)|n\rangle$$

$$= -qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\langle m|a_+|n\rangle + \langle m|a_-|n\rangle\right]$$

$$\langle \psi_m^0 | H' | \psi_n^0 \rangle = -qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\langle m | \sqrt{n+1}\,|n+1\rangle + \langle m | \sqrt{n}\,|n-1\rangle\right]$$

$$= -qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\sqrt{n+1}\,\underbrace{\langle m | n+1\rangle}_{\delta_{m,n+1}} + \sqrt{n}\,\underbrace{\langle m | n-1\rangle}_{\delta_{m,n-1}}\right]$$

$$= -qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right]$$

$$\Rightarrow E_n^2 = \sum_{m\neq n} \frac{|\langle \psi_m^0 | H' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n^0 - E_m^0 = \hbar\omega(n+1/2) - \hbar\omega(m+1/2)$$

$$E_n^0 - E_m^0 = (n-m)\hbar\omega$$

$$\Rightarrow E_n^2 = \sum_{m\neq n} \frac{\left|-qE\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right]\right|^2}{(n-m)\hbar\omega}$$

$$m \begin{cases} n+1 \\ n-1 \end{cases}$$

$$E_n^2 = \frac{q^2E^2\hbar}{2m\omega}\left[\frac{(n+1)}{\hbar\omega\,[n-(n+1)]} + \frac{n}{\hbar\omega\,[n-(n-1)]}\right]$$

$$E_n^2 = \frac{1}{2m}\left(\frac{qE}{\omega}\right)^2\left[\frac{n+1}{n-1} + \frac{n}{n+1}\right] = \frac{1}{2m}\left(\frac{qE}{\omega}\right)^2\left[-(n+1)+n\right]$$

$$E_n^2 = -\frac{1}{2m}\left(\frac{qE}{\omega}\right)^2 \;//$$

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