3.1. S_x, S_y, S_z ve S^2 spin operatörlerinin özdeğer ve özvektörlerini bulunuz.

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x}, \quad S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y}, \quad S_z = \frac{\hbar}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z} = \frac{\hbar}{2}\sigma_z$$

$$[S_x, S_y] = i\hbar S_z \qquad \sigma_j \sigma_k = \delta_{jk} + i\sum_{l} \varepsilon_{jkl}\sigma_l$$

$$[S_y, S_z] = i\hbar S_x$$

$\sigma_x, \sigma_y, \sigma_z$: Pauli spin matrisleri

$$[S_z, S_x] = i\hbar S_y$$

- Karakteristik denklem: $S_x|\chi_x\rangle = \lambda_x|\chi_x\rangle$

$$(S_x - \lambda_x I)|\chi_x\rangle = 0 \Rightarrow |S_x - \lambda_x I| = 0$$

$$|(S_x - \lambda_x I)| = \left|\frac{\hbar}{2}\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{S_x} - \lambda_x \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_{2\times2}}\right| = \left|\frac{\hbar}{2}\begin{pmatrix} -\lambda_x & 1 \\ 1 & -\lambda_x \end{pmatrix}\right| = \frac{\hbar}{2}\begin{vmatrix} -\lambda_x & 1 \\ 1 & -\lambda_x \end{vmatrix} = 0$$

$$\frac{\hbar}{2}\left[\lambda_x^2 - 1\right] = 0 \Rightarrow \underbrace{\lambda_{x1} = \frac{\hbar}{2}, \ \lambda_{x2} = -\frac{\hbar}{2}}_{\text{özdeğerler}}$$

$$\lambda_{1x} = \frac{\hbar}{2} \Rightarrow |\chi_{x1}\rangle$$

$$\lambda_{2x} = -\frac{\hbar}{2} \Rightarrow |\chi_{x2}\rangle$$

- $S_x|\chi_{x1}\rangle = \lambda_{x1}|\chi_{x1}\rangle \Rightarrow (S_x - \lambda_{x1} I)|\chi_{x1}\rangle = 0, \quad |\chi_{x1}\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\left[\underbrace{\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}}_{S_x} - \underbrace{\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & \frac{\hbar}{2} \end{pmatrix}}_{\lambda_{x1} I}\right]\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{|\chi_{x1}\rangle} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -\hbar/2 & \hbar/2 \\ \hbar/2 & -\hbar/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -\frac{\hbar}{2}a + \frac{\hbar}{2}b = 0 \Rightarrow a = b \Rightarrow \chi_{x1} = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\langle \chi_{x1}|\chi_{x1}\rangle = 1$$

$$a^2 (1\ 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

$$a = \frac{1}{\sqrt{2}}$$

$$\chi_{x1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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- $S_x|\chi_{x2}\rangle = \lambda_{x2}|\chi_{x2}\rangle \Rightarrow (S_x - \lambda_{x2} I)|\chi_{x2}\rangle = 0 \,, \quad |\chi_{x2}\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\Bigg[\underbrace{\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}}_{S_x} - \underbrace{\begin{pmatrix} -\hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}}_{\lambda_{x2} I}\Bigg] \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{|\chi_{x2}\rangle} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} +\hbar/2 & \hbar/2 \\ +\hbar/2 & \hbar/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{\hbar}{2}a + \frac{\hbar}{2}b = 0 \Rightarrow a = -b \Rightarrow |\chi_{x2}\rangle = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow |\chi_{x2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}_{//}$$

- Karakteristik denklem; $S_y|\chi_y\rangle = \lambda_y|\chi_y\rangle$

$$(S_y - \lambda_y)|\chi_y\rangle = 0 \Rightarrow |S_y - \lambda_y I| = 0$$

$$|S_y - \lambda_y I| = 0 \Rightarrow \Bigg|\underbrace{\begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix}}_{S_y} - \underbrace{\begin{pmatrix} \lambda_y & 0 \\ 0 & \lambda_y \end{pmatrix}}_{\lambda_y I}\Bigg| = 0 \Rightarrow \begin{vmatrix} -\lambda_y & -i\hbar/2 \\ i\hbar/2 & -\lambda_y \end{vmatrix} = 0$$

$$\Rightarrow \lambda_y^2 - \frac{\hbar^2}{4} \Rightarrow \lambda_{y1} = \frac{\hbar}{2}, \ \lambda_{y2} = -\frac{\hbar}{2}$$

$$\lambda_{y1} = \frac{\hbar}{2} \Rightarrow |\chi_{y1}\rangle$$

$$\lambda_{y2} = -\frac{\hbar}{2} \Rightarrow |\chi_{y2}\rangle$$

- $S_y|\chi_{y1}\rangle = \lambda_{y1}|\chi_{y1}\rangle \Rightarrow (S_y - \lambda_{y1} I)|\chi_{y1}\rangle = 0 \,, \quad |\chi_{y1}\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\Bigg[\underbrace{\begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix}}_{S_y} - \underbrace{\begin{pmatrix} \hbar/2 & 0 \\ 0 & +\hbar/2 \end{pmatrix}}_{\lambda_{y1} I}\Bigg] \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{|\chi_{y1}\rangle} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -\hbar/2 & -i\hbar/2 \\ i\hbar/2 & +i\hbar/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -\frac{\hbar}{2}a - i\frac{\hbar}{2}b = 0 \Rightarrow a = -ib \Rightarrow |\chi_{y1}\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ b \end{pmatrix} = b\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$|\chi_{y1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}_{//}$$

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- $S_y |\chi_{y2}\rangle = \lambda_{y2} |\chi_{y2}\rangle \Rightarrow (S_y - \lambda_{y2} I) |\chi_{y2}\rangle = 0 \ , \ |\chi_{y2}\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\left[\begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix} - \begin{pmatrix} -\hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}\right] \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \hbar/2 & -i\hbar/2 \\ i\hbar/2 & -\hbar/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -a\frac{\hbar}{2} + i\frac{\hbar}{2} b = 0 \Rightarrow a = ib \Rightarrow |\chi_{y2}\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ib \\ b \end{pmatrix} = b\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$|\chi_{y2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} //$$

- Karakteristik denklem; $S_z |\chi_z\rangle = \lambda_z |\chi_z\rangle \Rightarrow (S_z - \lambda_z I)|\chi_z\rangle = 0$

$$|S_z - \lambda_z I| = 0$$

$$|S_z - \lambda_z I| = 0 \Rightarrow \left| \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} - \begin{pmatrix} \lambda_z & 0 \\ 0 & \lambda_z \end{pmatrix} \right| = 0 \Rightarrow \begin{matrix} \lambda_{z1} = \hbar/2 \to |\chi_{z1}\rangle \\ \lambda_{z2} = -\hbar/2 \to |\chi_{z2}\rangle \end{matrix}$$

$$|\chi_{z1}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} // \quad |\chi_{z2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} //$$

- $S^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda_{S^2} = 3\hbar^2/4 \ , \quad |\chi_{Sz1}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\chi_{Sz2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3.2. Spini 1/2 olan bir parçacığın;

$$|\chi\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1+i\\2\end{pmatrix}$$

durumunda olduğunu varsayalım. S_z ve S_x'i ölçerseniz $+\hbar/2$ ve $-\hbar/2$ bulma olasılığının nedir?

- $|S_z=\hbar/2\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |S_z=-\hbar/2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

- $P(S_z=\hbar/2) = \left|\langle S_z=\hbar/2|\chi\rangle\right|^2 = \left|\frac{1}{\sqrt{6}}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1+i\\2\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{6}}(1+i+0)\right|^2$

$P(S_z=\hbar/2) = \frac{1}{6}|1+i|^2 = \frac{1}{6}(1-i)(1+i) = 1/3$

$P(S_z=\hbar/2) = 1/3$ //

- $P(S_z=-\hbar/2) = \left|\langle S_z=-\hbar/2|\chi\rangle\right|^2 = \left|\frac{1}{\sqrt{6}}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1+i\\2\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{6}}(0+2)\right|^2 = \frac{4}{6}$

$P(S_z=-\hbar/2) = \frac{2}{3}$ // $\qquad P(S_z=\hbar/2) + P(S_z=-\hbar/2) = \frac{1}{3}+\frac{2}{3} = 1$ //

- $|S_x=\hbar/2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad |S_x=-\hbar/2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

- $P(S_x=\hbar/2) = \left|\langle S_z=\hbar/2|\chi\rangle\right|^2 = \left|\frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1+i\\2\end{pmatrix}\right|^2$

$= \left|\frac{1}{\sqrt{12}}(1+i+2)\right|^2 = \left|\frac{3+i}{\sqrt{12}}\right|^2 = \frac{1}{12}(3-i)(3+i) = 5/6$

$P(S_x=\hbar/2) = 5/6$ //

$P(S_x=-\hbar/2) = +1/6$ //

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3. Bir elektron

$$|\chi\rangle = A\begin{pmatrix}3i\\4\end{pmatrix}$$

ile verilen bir spin durumundadır.

a) Normalleme sabiti A'yı belirleyin

b) S_x, S_y, S_z'nin beklenen değerlerini bulun.

c) σ_{S_x}, σ_{S_y} ve σ_{S_z} belirsizliklerini bulun.

a: $\langle\chi|\chi\rangle = 1 \Rightarrow \underbrace{A^*\begin{pmatrix}-3i & 4\end{pmatrix}}_{\langle\chi|}\underbrace{A\begin{pmatrix}3i\\4\end{pmatrix}}_{|\chi\rangle} = 1 \Rightarrow |A|^2\left[+9+16\right] = 1$

$$A = \frac{1}{5}$$

$$|\chi\rangle = \frac{1}{5}\begin{pmatrix}3i\\4\end{pmatrix}$$

b: • $\langle S_x\rangle = \langle\chi|S_x|\chi\rangle = \frac{1}{25}\left[\underbrace{\begin{pmatrix}-3i & 4\end{pmatrix}}_{\langle\chi|}\underbrace{\begin{pmatrix}0 & \hbar/2\\ \hbar/2 & 0\end{pmatrix}}_{S_x}\underbrace{\begin{pmatrix}3i\\4\end{pmatrix}}_{|\chi\rangle}\right]$

$$\langle S_x\rangle = \frac{1}{25}\left[\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0+2\hbar\\ \frac{3i\hbar}{2}+0\end{pmatrix}\right] = \frac{1}{25}\left[-3i(2\hbar)+4\left(\frac{3i\hbar}{2}\right)\right] = \frac{1}{25}\left(-6i\hbar+6i\hbar\right)$$

$$\langle S_x\rangle = 0 \,//$$

• $\langle S_y\rangle = \langle\chi|S_y|\chi\rangle = \frac{1}{25}\left[\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & -i\hbar/2\\ i\hbar/2 & 0\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}\right]$

$$= \frac{1}{25}\left[\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0-2i\hbar\\ -\frac{3\hbar}{2}+0\end{pmatrix}\right] = \frac{1}{25}\left[-6\hbar-6\hbar\right] = -\frac{12\hbar}{25}$$

$$\langle S_y\rangle = -\frac{12\hbar}{25}\,//$$

• $\langle S_z\rangle = \langle\chi|S_z|\chi\rangle = \frac{1}{25}\left[\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}\hbar/2 & 0\\ 0 & -\hbar/2\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}\right] \Rightarrow \langle S_z\rangle = -\frac{7\hbar}{50}$

$$\langle S_z\rangle = -\frac{7\hbar}{50}\,//$$

c:

$$\sigma_{S_x} = \left[\langle S_x^2 \rangle - \underbrace{\langle S_x \rangle^2}_{0} \right]^{1/2} = \left[\langle S_x^2 \rangle \right]^{1/2}$$

$$S_x^2 = S_x S_x = \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} = \begin{pmatrix} 0 + \hbar^2/4 & 0+0 \\ 0+0 & \frac{\hbar^2}{4}+0 \end{pmatrix} = \frac{\hbar^2}{4} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}$$

$\sigma_i \sigma_i = I$

$\sigma_x \sigma_x = I$

$$\langle S_x^2 \rangle = \langle x | S_x^2 | x \rangle = \hbar^2/4$$

Spini 1/2 olan parçacık için; $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \hbar^2/4$

- $\sigma_{S_x} = \left[\langle S_x^2 \rangle \right]^{1/2} = \left[\frac{\hbar^2}{4} \right]^{1/2} = \hbar/2$ //

- $\sigma_{S_y} = \left[\langle S_y^2 \rangle - \underbrace{\langle S_y \rangle^2}_{-12\hbar/25} \right]^{1/2} = \sqrt{\frac{\hbar^2}{4} - \left(-\frac{12\hbar}{25} \right)^2}$ //

- $\sigma_{S_z} = \left[\langle S_z^2 \rangle - \langle S_z \rangle^2 \right]^{1/2} = \sqrt{\frac{\hbar^2}{4} - \left(\frac{7\hbar}{50} \right)^2}$ //

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3.4. Spini 1 olan bir parçacık için S_x, S_y ve S_z spin matrislerini kurun.

$$S_+ = S_x + i S_y \qquad S_x = \frac{1}{2}(S_+ + S_-)$$
$$S_y = S_x - i S_y \qquad S_y = \frac{1}{2i}(S_+ - S_-)$$

$s = 1 \Rightarrow$
$m_s : -1, 0, 1$

$$S_\pm |s, m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s \pm 1)}\, |s, m_s \pm 1\rangle$$
$$S |s, m_s\rangle = \hbar \sqrt{s(s+1)}\, |s, m_s\rangle$$
$$S_z |s, m_s\rangle = m_s \hbar\, |s, m_s\rangle$$

$$(S_z)_{m_s m_s'} = \langle s, m_s' | S_z | s, m_s\rangle = \langle s, m_s' | \hbar m_s | s, m_s\rangle$$
$$= \hbar m_s \langle s, m_s' | s, m_s'\rangle$$
$$= \hbar m_s\, \delta_{mm'}$$

$s = 1 \Rightarrow m_s = -1, 0, 1$

- $$S_z = \begin{array}{c} \\ m_s=1 \\ m_s=0 \\ m_s=-1 \end{array} \begin{array}{c} \begin{array}{ccc} m_s=1 & m_s=0 & m_s=-1 \end{array} \\ \begin{pmatrix} \delta_{11}\hbar & \cancel{\delta_{10}}^{0} & \cancel{\delta_{1-1}}^{0} \\ \cancel{\delta_{01}}^{0} & \delta_{00}0\hbar & \cancel{\delta_{0-1}}^{0} \\ \cancel{\delta_{-11}}^{0} & \cancel{\delta_{-10}} & \delta_{-1-1}(-\hbar) \end{pmatrix} \end{array} \Rightarrow S_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} //$$

- $$(S_+)_{m_s m_s'} = \langle s, m_s' | S_+ | s, m_s\rangle = \langle s, m_s' | \hbar\sqrt{s(s+1) - m_s(m_s+1)}\, | s, m_s+1\rangle$$
$$= \hbar \sqrt{2 - m_s(m_s+1)}\, \langle s, m_s' | s, m_s+1\rangle$$
$$= \hbar \sqrt{2 - m_s(m_s+1)}\, \delta_{m_s' m_s+1}$$

$$S_+ = \begin{array}{c} m_s = \\ 1 \\ 0 \\ -1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 0 & -1 \end{array} \\ \begin{pmatrix} 0 & \hbar\sqrt{2} & 0 \\ 0 & 0 & \hbar\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \end{array} = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad (S_+)^\dagger = S_-$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\Rightarrow S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} //, \qquad S_y = -\frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} //$$

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