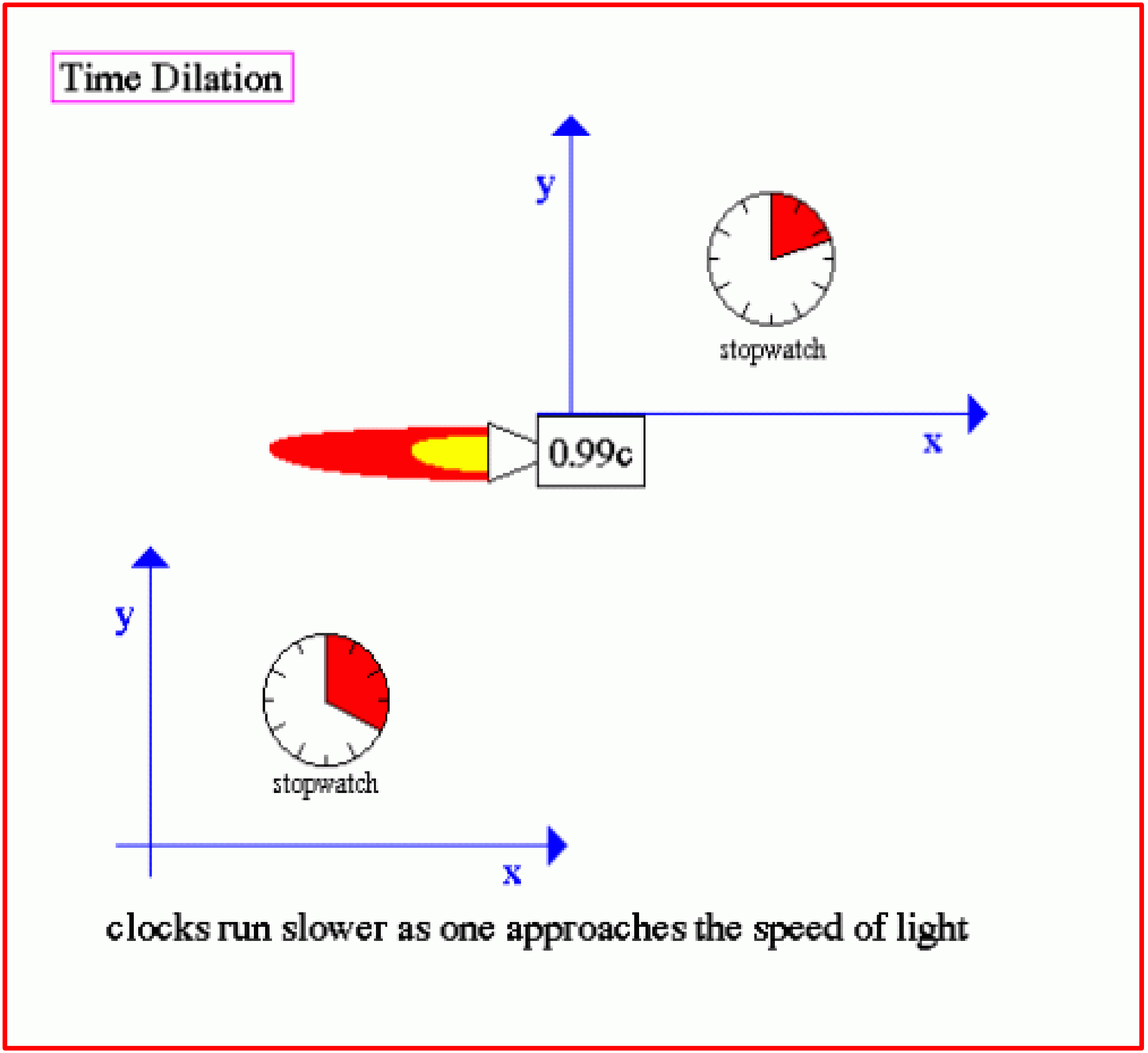
Special Theory of Relativity

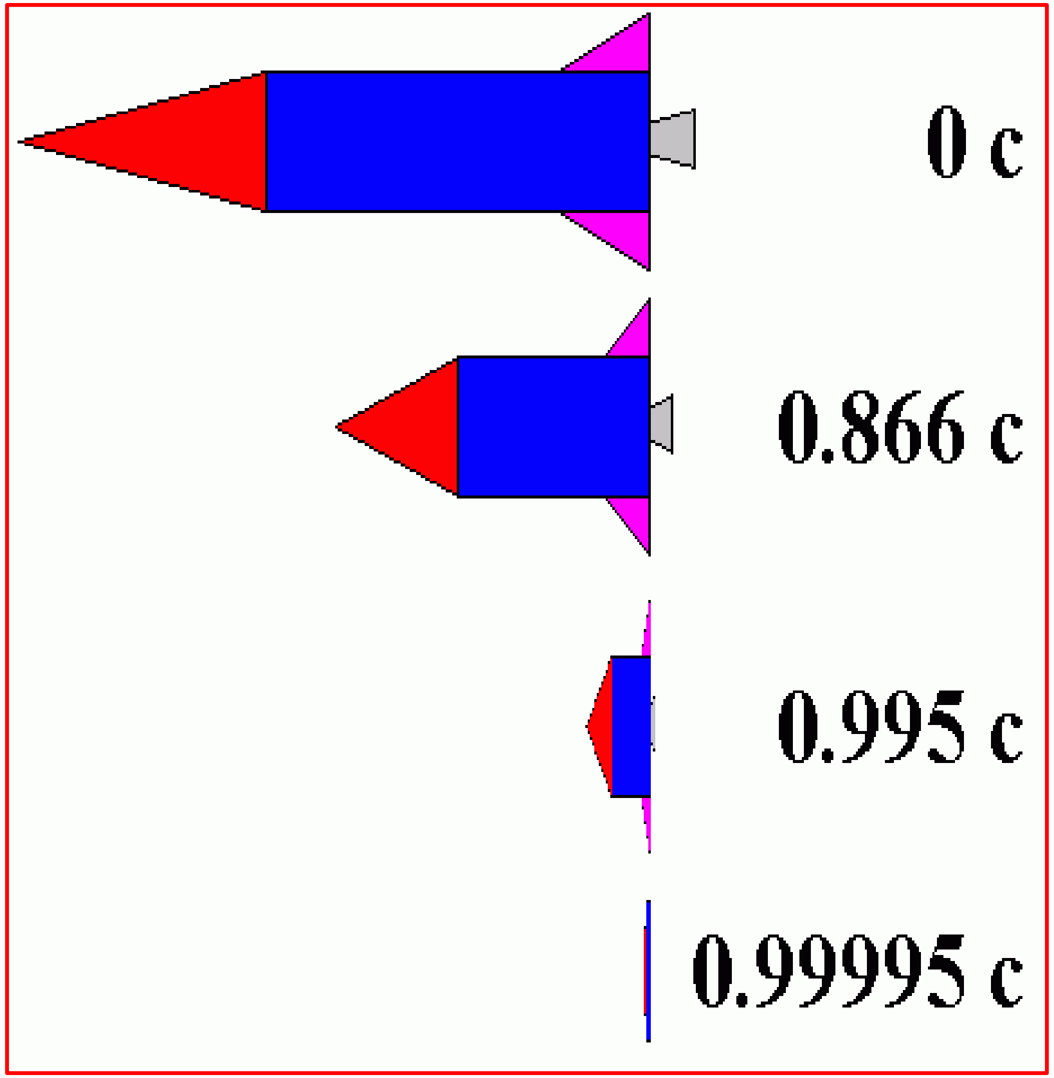
LECTURE 2

Special Relativity

- The theory of special relativity assumes that the **velocity of light c** is a universal speed limit.
- There is no universal clock in spacetime. Which means that **time is *relative* and not absolute.**
- Time is a fourth dimension in addition to the three spatial dimensions that we move around in.
- **The implications of special relativity are that moving clocks slow down and that moving objects shrink in the direction of motion.**



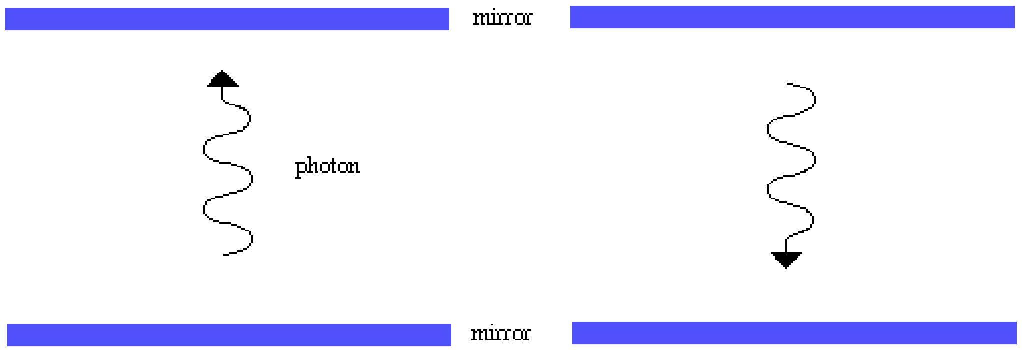
Length Contraction



Observed Length of an Object: Observed length of an object at rest and at different speeds

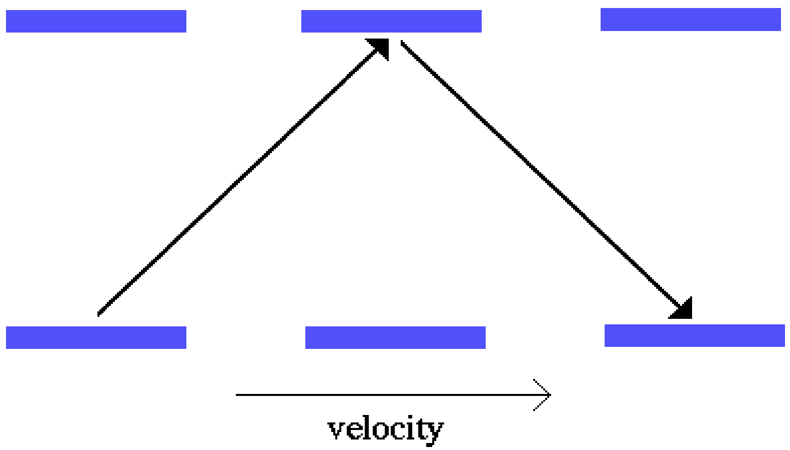
Two events that occur at **different times** for one person can be simultaneous for a second person in motion relative to the first.

Photon Clock



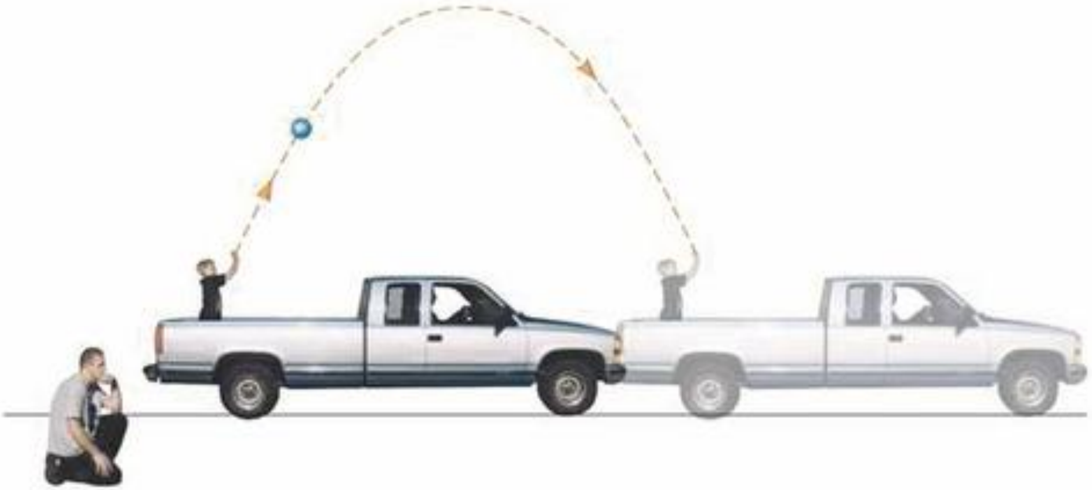
A photon clock is a device that consists of two mirrors and a photon passing between them. Each time the photon strikes a mirror corresponds to one 'tick' of the clock.

To understand time dilation, consider a moving photon clock. Since special relativity says that a photon must travel at a constant speed (c), then the path of the photon for a moving clock is longer than a clock at rest.



Therefore, the time between 'ticks' is longer for a moving clock, as seen from a rest frame.

(a)



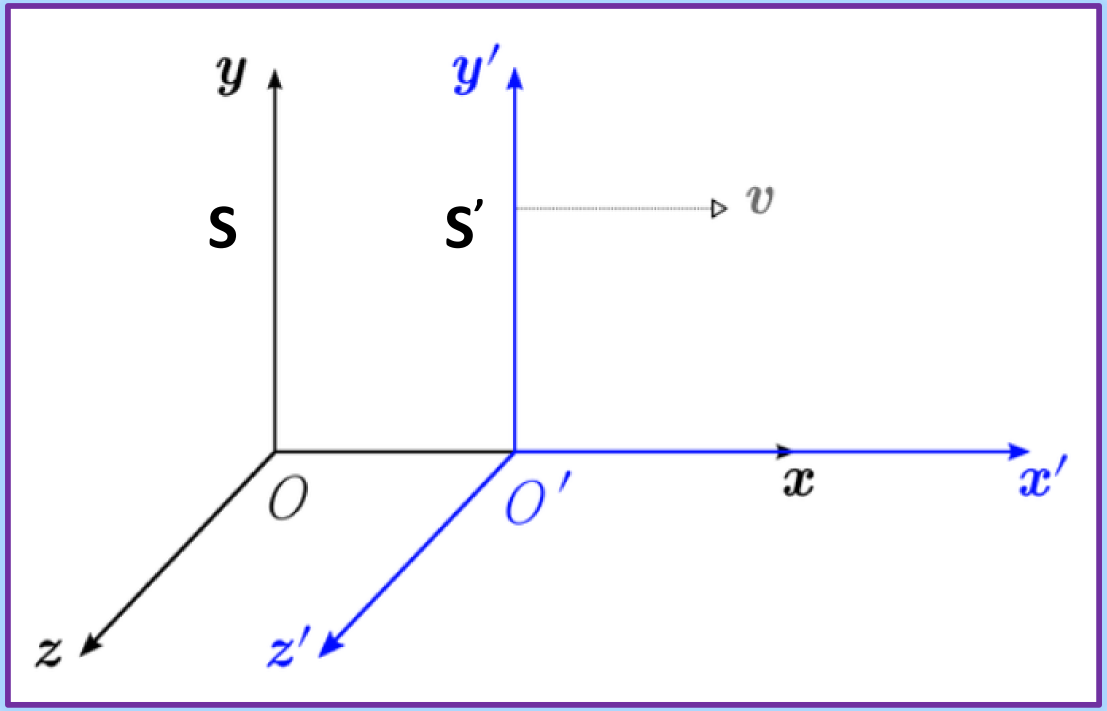
(b)

- **The observer in the truck throws a ball straight up**
  - It appears to move in a vertical path
  - The law of gravity and equations of motion under uniform acceleration are obeyed
- **There is a stationary observer on the ground**
  - Views the path of the ball thrown to be a parabola
  - The ball has a velocity to the right equal to the velocity of the truck

- Special relativity requires an understanding of **Lorentz transformations, time dilation, and Fitzgerald-Lorentz contraction.**
- The Minkowski diagram provides a geometric interpretation of events in spacetime.
- A sample diagram shows how two inertial frames in relative motion exhibit time dilations and contractions in both directions.
- Different frames of reference are in complete agreement about relative velocity, events in spacetime, and the laws of physics.

Galilean transformation of coordinate

- Consider two coordinate systems S and S′ moving relative to each other in the x-direction.
- Specifically, coordinate system S′ is moving to the right in the direction of the positive x-axis at constant velocity $v_{S'/S}$ relative to S.
- Alternatively, coordinate system S is moving to the left at constant velocity $v_{S/S'}$ relative to S′ so that $v_{S/S'} = -v_{S'/S}$.



- Relative motion occurs only along the x-axis; y and z coordinates coincide in S and S′.

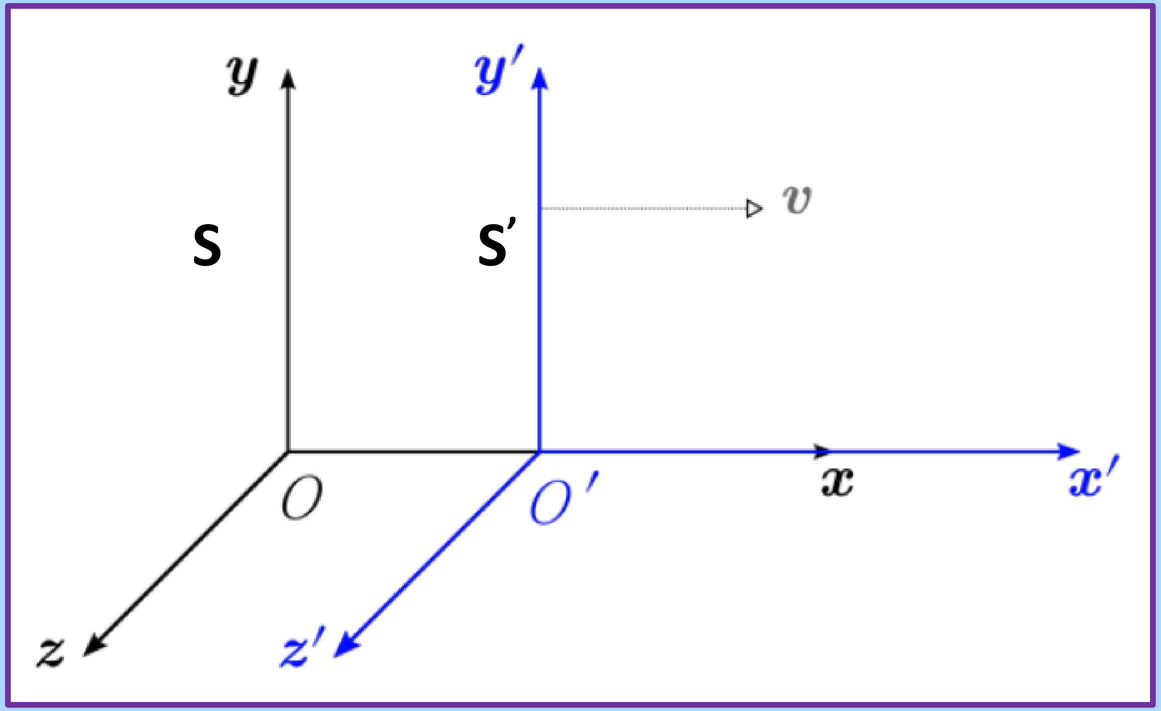
- Initial conditions at t = 0 are x = x′= 0.

- $v_{S'/S}$ is calculated by setting x'= 0 (the location of the origin of S′) :

$$v_{S'/S} = \frac{x}{t} = v \qquad (\text{for } x' = 0)$$

- The **Galilean coordinate transformations** are:

$$x' = x - vt$$
$$x = x' + vt$$
$$y = y'$$
$$z = z'$$
$$t = t'$$



Velocities are additive:

$$u'_x = u_x - v$$
$$u_x = u'_x + v$$

Acceleration is an invariant :

$$a'_x = a_x$$

Galilean Space-Time Transformation

Coordinates: $$\boxed{x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t}$$

Velocities: $$\boxed{u_{x'} = \frac{dx'}{dt'} = \frac{dx}{dt} - v = u_x - v \qquad u_{y'} = u_y \qquad u_{z'} = u_z}$$

Accelerations: $$\boxed{a_{x'} = \frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} = a_x \qquad a_{y'} = a_y \qquad a_{z'} = a_z}$$

Newton's Laws involving accelerations are invariant with respect to Galilean transformations!

Einstein’s Postulates

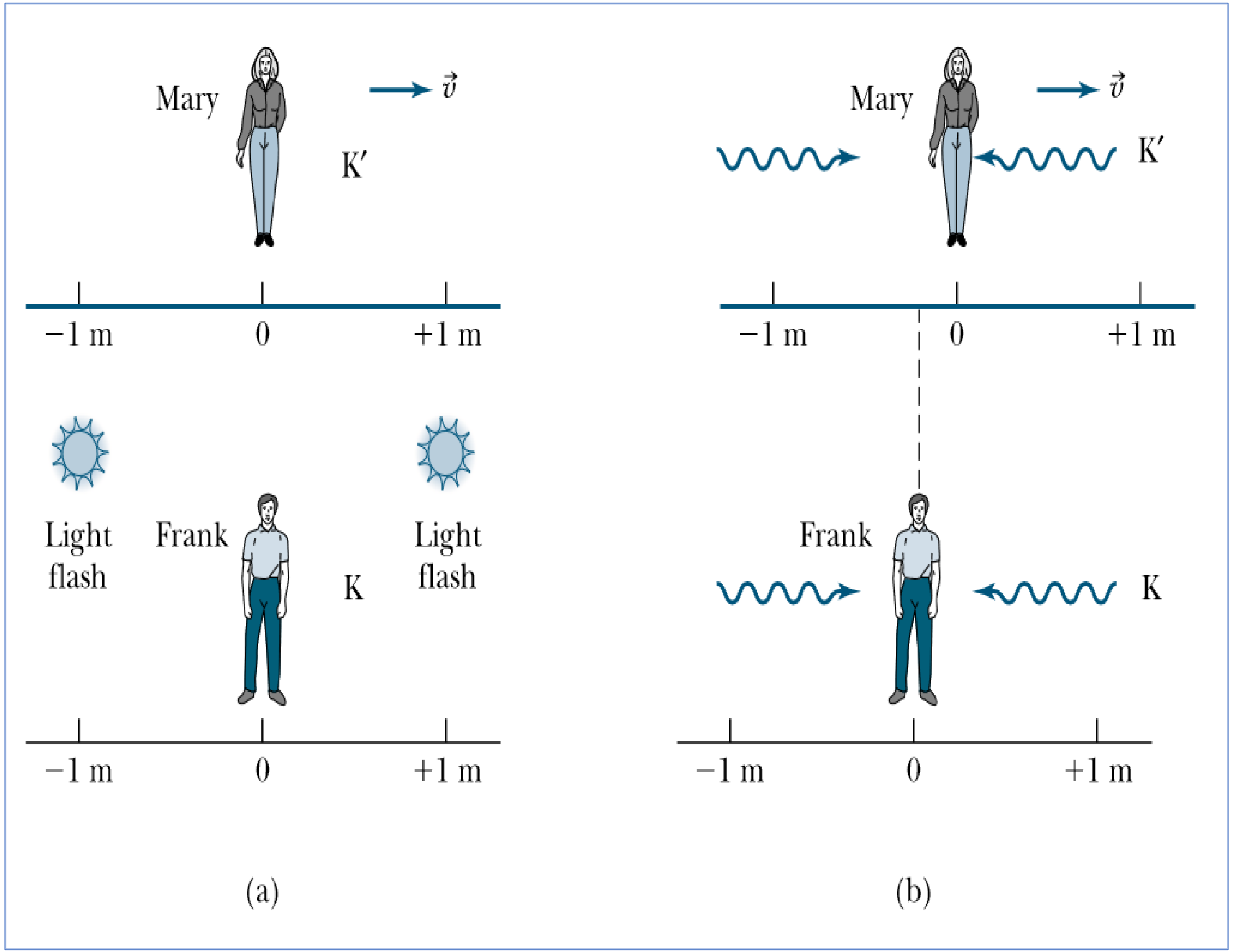
- At the turn of the twentieth century, the Michelson-Morley experiment had laid to rest the idea of finding a preferred inertial system for Maxwell’s equations.
- Because the Galilean transformation, which worked for the laws of mechanics, was **invalid for** Maxwell’s equations.
- Einstein started to think to **form of Maxwell’s equations** in moving inertial systems, and in 1905, he published his startling proposal about the principle of relativity.
- **He believed that Maxwell’s equations must be valid in all inertial frames.**

- Einstein was able to bring together seemingly inconsistent results concerning the **laws of mechanics and electromagnetism** with two postulates :

1. **The principle of relativity:** The laws of physics are the same in all inertial systems. There is no way to detect absolute motion, and no preferred inertial system exists.
2. **The constancy of the speed of light:** Observers in all inertial systems measure the same value for the speed of light in a vacuum.

- The **first postulate** indicates that the laws of physics are the same in all coordinate systems moving with **uniform relative motion to each other**.
- Einstein showed that **postulate 2** actually follows from the first one.
- Although Newton's principle referred only to the laws of mechanics, Einstein expanded it to include all laws of physics—including those of electromagnetism.
- We can now **modify** our previous definition of inertial frames of reference to be those frames of reference in which all the laws of physics are valid.

- Einstein's solution requires us to take a careful look at **time**.



The problem of **simultaneity**.

Flashbulbs positioned in system K at one meter on either side of Frank go off simultaneously in (a).
Frank indeed sees both flashes **simultaneously** in (b).

However, Mary, at rest in system K'moving to the right with speed v, does **not** see the flashes **simultaneously** despite the fact that she was alongside Frank when the flashbulbs went off.

During the finite time it took light to travel the one meter, Mary has moved slightly, as shown in exaggerated form in (b).

Einstein’s Postulates

- We must be careful when comparing the **same event** in two systems moving with respect to one another.
- **Time comparison can be accomplished** by sending light signals from one observer to another, but this information can travel only as fast as the finite speed of light.
- It is best if each system has its own observers with clocks that are synchronized. How can we do this?
- **We place observers with clocks throughout a given system**.

Einstein’s Postulates

- If, when we bring all the clocks together at one spot at rest, all the clocks agree, then the clocks are said to be **synchronized**. But to move the clocks relative to each other to reposition them this might affect the synchronization.
- So a better way would be to flash a bulb half way between each pair of clocks at rest and make sure the pulses arrive simultaneously at each clock.
- **This will require many measurements, but it is a safe way to synchronize the clocks.**
- We can determine the time of an event occurring far away from us by having a colleague at the event, with a clock fixed at rest, measure the time of the particular event, and send us the results, for example, by telephone or even by mail.
- If we need to check our clocks, we can always send light signals to each other over known distances at some predetermined time.

Einstein's Postulates

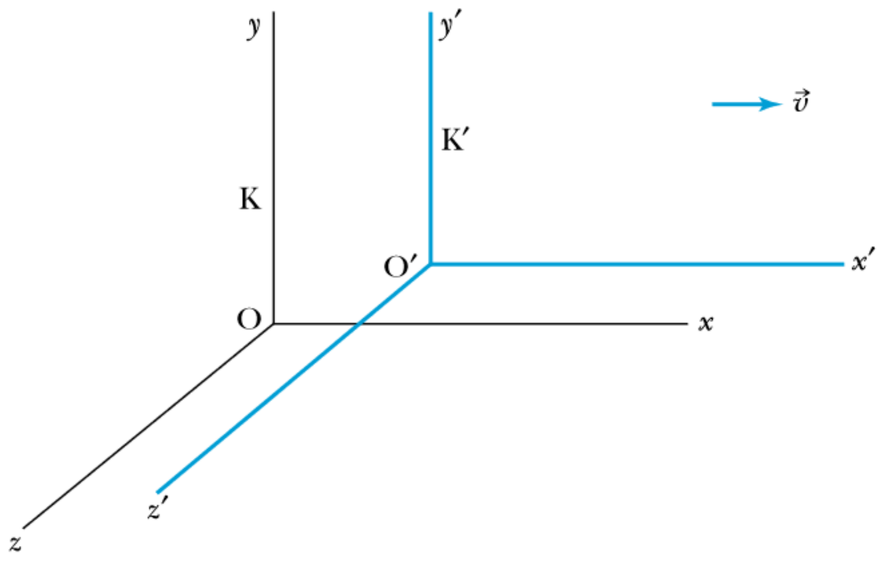
- Now we derive the correct transformation, called the **Lorentz transformation**, that makes the laws of physics **invariant between inertial frames of reference**. We use the coordinate systems described by Figure.
- At ***t = t'* =0**, the origins of the two coordinate systems are coincident, and the system K' is traveling along the *x* and *x'* axes.
- For this special case, the **Lorentz transformation equations** are:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - (vx/c^2)}{\sqrt{1 - v^2/c^2}}$$



Einstein's Postulates

- Relativistic factor: We commonly use the symbols **β and the *relativistic factor* γ** to represent two longer expressions:

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

which allows the Lorentz transformation equations to be **rewritten in compact form** as:

$$x' = \gamma(x - \beta ct)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - \beta x/c)$$

Note that **γ** ≥ 1 (**γ** = 1 when $\boldsymbol{v}$ =0).

The Lorentz Transformation

- We use **Einstein's two postulates** to find a transformation between inertial frames of reference such that all the physical laws, including Newton's laws of mechanics and Maxwell's electrodynamics equations, will have the same form.
- We use the fixed system K and moving system K′.
- At ***t = t' =* 0** the origins and axes of both systems are coincident, and system K' is moving to the right along the *x* axis.
- A flashbulb **goes off** at the origins when $t = t' = 0$.
- According to postulate 2, **the speed of light will be c in both systems,** and the wavefronts observed in both systems must be spherical and described by

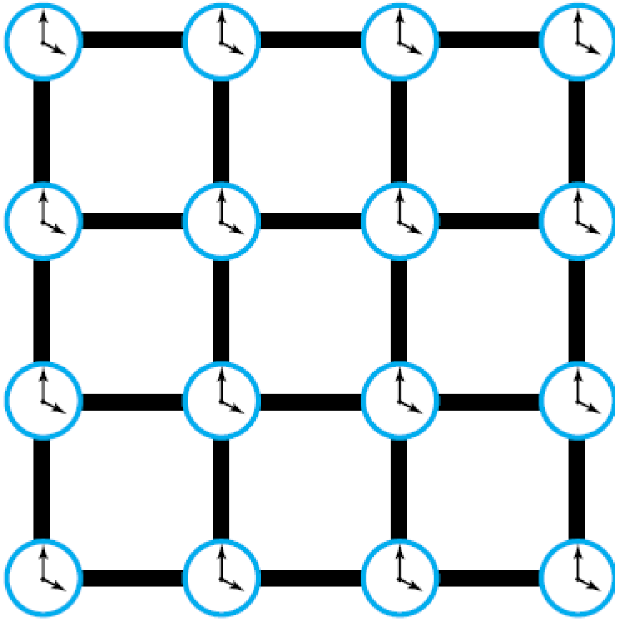
$$\boxed{\begin{aligned} x^2 + y^2 + z^2 &= c^2 t^2 \\ x'^2 + y'^2 + z'^2 &= c^2 t'^2 \end{aligned}}$$

The Lorentz Transformation

- These two equations are **inconsistent** with a Galilean transformation because a wavefront can be spherical in only one system when the second is moving at speed v with respect to the first.
- The Lorentz transformation *requires* both systems to have a spherical wavefront centered on each system's origin.
- **Another clear break with Galilean and Newtonian physics is that we do not assume that *t* = *t'*.**
- Each system **must have its own clocks and metersticks** as indicated in a two-dimensional system in Figure.
- Because the systems move only **along their *x* axes**, observers in both systems agree by direct observation that

$$y' = y$$

$$z' = z$$



In order to make sure accurate event measurements can be obtained, **synchronized clocks** and **uniform measuring sticks** are placed throughout a system.

- We know that the Galilean transformation

$$x' = x - vt$$

is **incorrect**, but what is the correct transformation?

- We require a **linear transformation** so that each event in system K corresponds to one, and only one, event in system K'.

- The **simplest *linear* transformation** is of the form

$$x' = \gamma(x - vt)$$

The Lorentz Transformation

- We will see if such a transformation **answers**.
- The parameter **γ** cannot depend on ***x*** or ***t*** because the transformation must be linear.
- The parameter **γ** must be close to **1** for ***v* << *c*** in order for Newton's laws of mechanics to be valid for most of our measurements.
- We can use similar arguments from the standpoint of an observer stationed in system K' to obtain an equation

$$x = \gamma'(x' + vt')$$

- Because postulate 1 requires that **the laws of physics be the same in both reference systems**, we demand that $\gamma' = \gamma$.

- Notice that the only difference between

$$x' = \gamma(x - vt)$$

and

$$x = \gamma'(x' + vt')$$

other than the primed and unprimed quantities being switched is that $v \rightarrow -v$, which is reasonable because according to the observer in each system, the other observer is moving either **forward or backward**.

The Lorentz Transformation

- According to **postulate 2**, the **speed of light is *c* in both systems**.

- Therefore, in each system the **wavefront of the flashbulb light pulse** along the respective *x* axes must be described by

$$x = ct \text{ and } x' = ct',$$

which we substitute into Equations to obtain

$$ct' = \gamma(ct - vt) \qquad ct = \gamma'(ct' - vt')$$

The Lorentz Transformation

We divide each of these equations by c and obtain

$$t' = \gamma t\left(1 - \frac{v}{c}\right)$$

and

$$t = \gamma t'\left(1 + \frac{v}{c}\right)$$

We substitute the value of t from second equation into the first equation.

$$t' = \gamma^2 t'\left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right)$$

We solve this equation for γ^2 and obtain

$$\gamma^2 = \frac{1}{1 - v^2/c^2}$$

or

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The Lorentz Transformation

In order to find a transformation for time t', we rewrite the equation as

$$t' = \gamma\left(t - \frac{vt}{c}\right)$$

We substitute $t = x/c$ for the light pulse and find

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \frac{t - vx/c^2}{\sqrt{1-\beta^2}}$$

We are now able to write the complete Lorentz transformations as

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1-\beta^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (vx/c^2)}{\sqrt{1-\beta^2}} \end{aligned} \tag{2.17}$$

The Lorentz Transformation

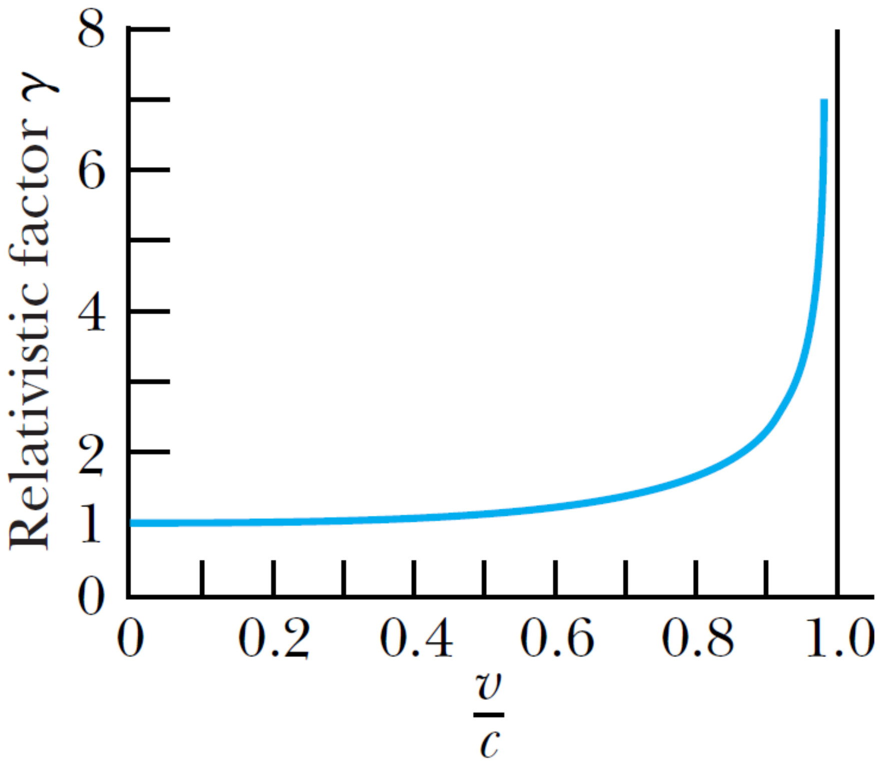
The inverse transformation equations are obtained by replacing v by $-v$ as discussed previously and by exchanging the primed and unprimed quantities.

$$\begin{aligned}
x &= \frac{x' + vt'}{\sqrt{1-\beta^2}} \\
y &= y' \\
z &= z' \\
t &= \frac{t' + (vx'/c^2)}{\sqrt{1-\beta^2}}
\end{aligned} \tag{2.18}$$

The Lorentz Transformation

- **Notice that these Lorents transformation Equations both reduce to the Galilean transformation when v << c.**
- It is only for **speeds that approach the speed of light** that the **Lorentz** transformation equations become significantly **different** from the **Galilean** equations.
- In our studies of **mechanics** we normally do not consider such high speeds, and our previous results probably require no corrections.
- The laws of mechanics credited to Newton are still valid over the region of their applicability.
- Even for a speed as high as the Earth orbiting about the sun, 30 km/s, the value of the relativistic factor γ is **1.000000005**.

- We show a plot of the relativistic parameter $\boldsymbol{\gamma}$ **versus speed** in Figure.
- As a rule of thumb, we should consider using the relativistic equations when $v/c > 0.1$ ($\gamma \approx 1.005$).



A plot of the relativistic factor γ as a function of speed v/c, showing that $\boldsymbol{\gamma}$ **becomes large quickly as v approaches c**.

The Lorentz Transformation

- The implications of Lorentz transformation.
- **The linear transformation equations ensure that a single event in one system is described by a single event in another inertial system.**
- However, **space and time** are not separate.
- In order to express the position of x in system K’, we must use both x’ and t’.

- We have also found that the Lorentz transformation does not allow a speed greater than c; the relativistic factor γ becomes **imaginary** in this case.

- For relatively **low velocities** where the relativity factor $\boldsymbol{\gamma}$ **is close to one**, the binomial expansion can be used to evaluate the small relativistic corrections.

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots$$

$$\gamma = 1 + \frac{\beta^2}{2} + \frac{3\beta^4}{8} + \ldots$$

Time Dilation

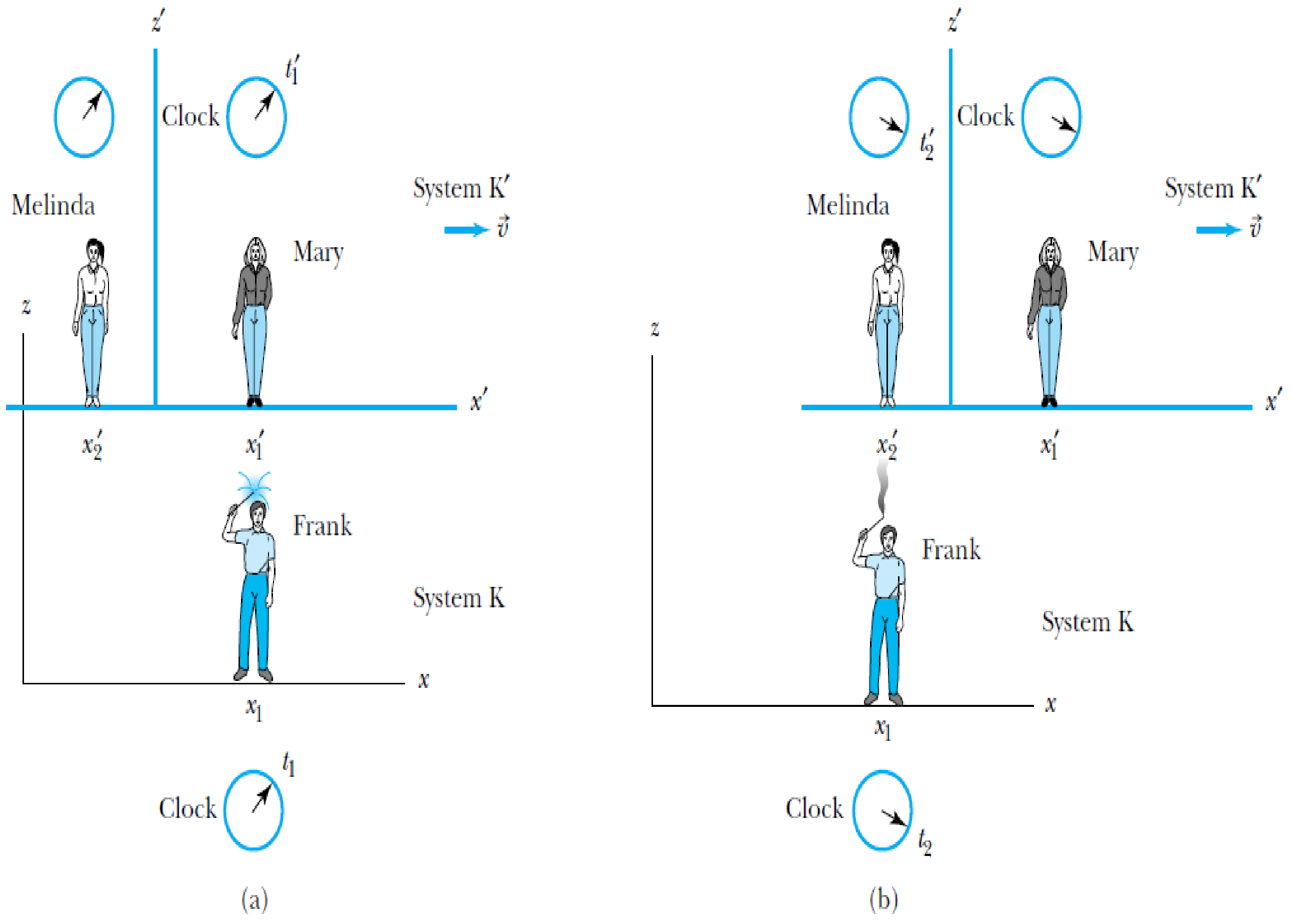
- The Lorentz transformations have immediate consequences with respect to time and length measurements made by observers in different inertial frames.

- We shall consider **time and length** measurements **separately** and then see how they are related to one another.

- Consider again our two systems K and K' with system K fixed and system K' moving along the *x* axis with velocity $\vec{v}$ as shown in the next Figure.

Time Dilation

- Frank lights a sparkler at position x_1 in system K.
- A clock placed beside the sparkler indicates the time to be t_1 when the sparkler is lit and t_2 when the sparkler goes out in next Figure (b).
- The sparkler burns for time T_0, where $T_0 = t_2 - t_1$.
- The time difference between two events occurring at the same position in a system as measured by a clock at rest in the system is called the proper time.
- We use the subscript zero on the **time difference** T_0 to denote the **proper time**.



Frank measures the proper time for the time interval that a sparkler stays lit.
His clock is at the same position in system K when the sparkler is lit in (a) and when it goes out in (b).
Mary, in the moving system K', is beside the sparkler at position x_1' when it is lit in (a), but by the time it goes out in (b), she has moved away.
Melinda, at position x_2', measures the time in system K' when the sparkler goes out in (b).

Time Dilation

- Now what is the time as determined by **Mary** who is passing by (but at rest in her own system K')?
- All the clocks in both systems have been **synchronized when the systems are at rest** with respect to one another.
- The two events (**sparkler lit and then going out**) do not occur at **the same place** according to **Mary.**
- She is beside the sparkler when it is lit, but she has moved **far away from the sparkler** when it goes out (Figure b).
- **Melinda**, also at rest in system K', is beside the sparkler when it goes out.
- **Mary and Melinda measure the two times for the sparkler to be lit and to go out in system K' as times t'_1 and t'_2.**
- The Lorentz transformation relates these times to those measured in system K as

$$t'_2 - t'_1 = \frac{(t_2 - t_1) - (v/c^2)(x_2 - x_1)}{\sqrt{1 - v^2/c^2}}$$

Time Dilation

- In system K the clock is fixed at x_1, so $\boldsymbol{x_2 - x_1 = 0}$; that is, the two events **occur at the same position**.
- The time $t_2 - t_1$ is the proper time T_0, and we denote the time difference $t'_2 - t'_1 = T'$ as measured in the moving system K:

$$T' = \frac{T_0}{\sqrt{1 - v^2/c^2}} = \gamma T_0$$

- **Thus the time interval measured in the moving system K' is greater than the time interval measured in system K where the sparkler is at rest.**

- This effect is known as **time dilation** and is a **direct result of Einstein's two postulates**.
- The time measured by Mary and Melinda in their system K' for the time difference was **greater than T_0** by the relativistic factor $\boldsymbol{\gamma}$ ($\gamma > 1$).
- The two events, sparkler being lit and then going out, did not occur at the same position ($x'_2 \neq x'_1$) in system K' (Figure b).
- **This result occurs because of the absence of simultaneity**.
- **The events do not occur at the same space and time coordinates in the two systems.**
- It requires three clocks to perform the measurement: one in system K and two in system K'.

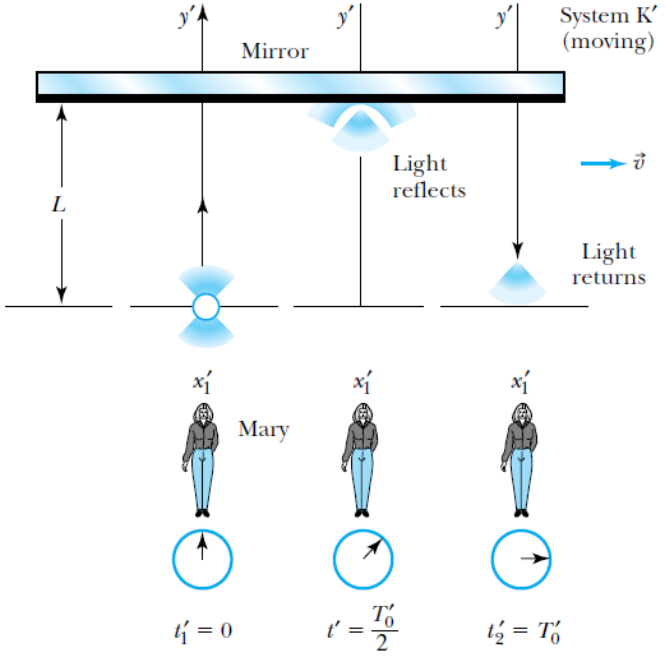
Time Dilation

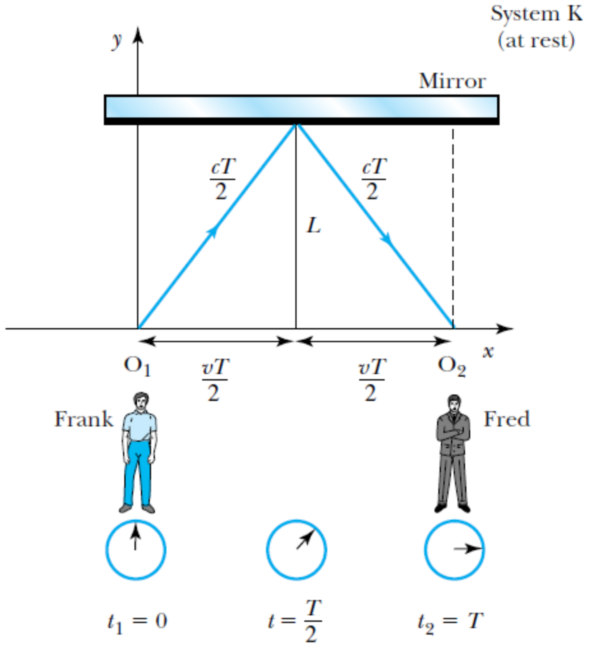
- The time dilation result is often interpreted by saying that ***moving clocks run slow*** by the factor γ^{-1}, and sometimes this is a useful way to remember the effect.
- The moving clock in this case can be **any kind of clock**.
- It can be the time a sparkler stays lit, the time between heartbeats, the time between ticks of a clock, or the time spent in a class lecture.
- In all cases, the actual **time interval on a moving clock is greater than the proper time as measured on a clock at rest**.

- The proper time is always the smallest possible time interval between two events.
- **Each person will claim the clock in the other (moving) system is running slow.**
- If **Mary** had a sparkler in her system K' at rest, **Frank** (fixed in system K) **would also measure a longer time interval on his clock in system K** because the sparkler would be moving with respect to his system.
- This is consistent with the earlier result. In this case $T > T_0'$.
- The proper time is always the shortest time interval, and we find that the clock in Mary's system K' is "**running slow**."

Time Dilation

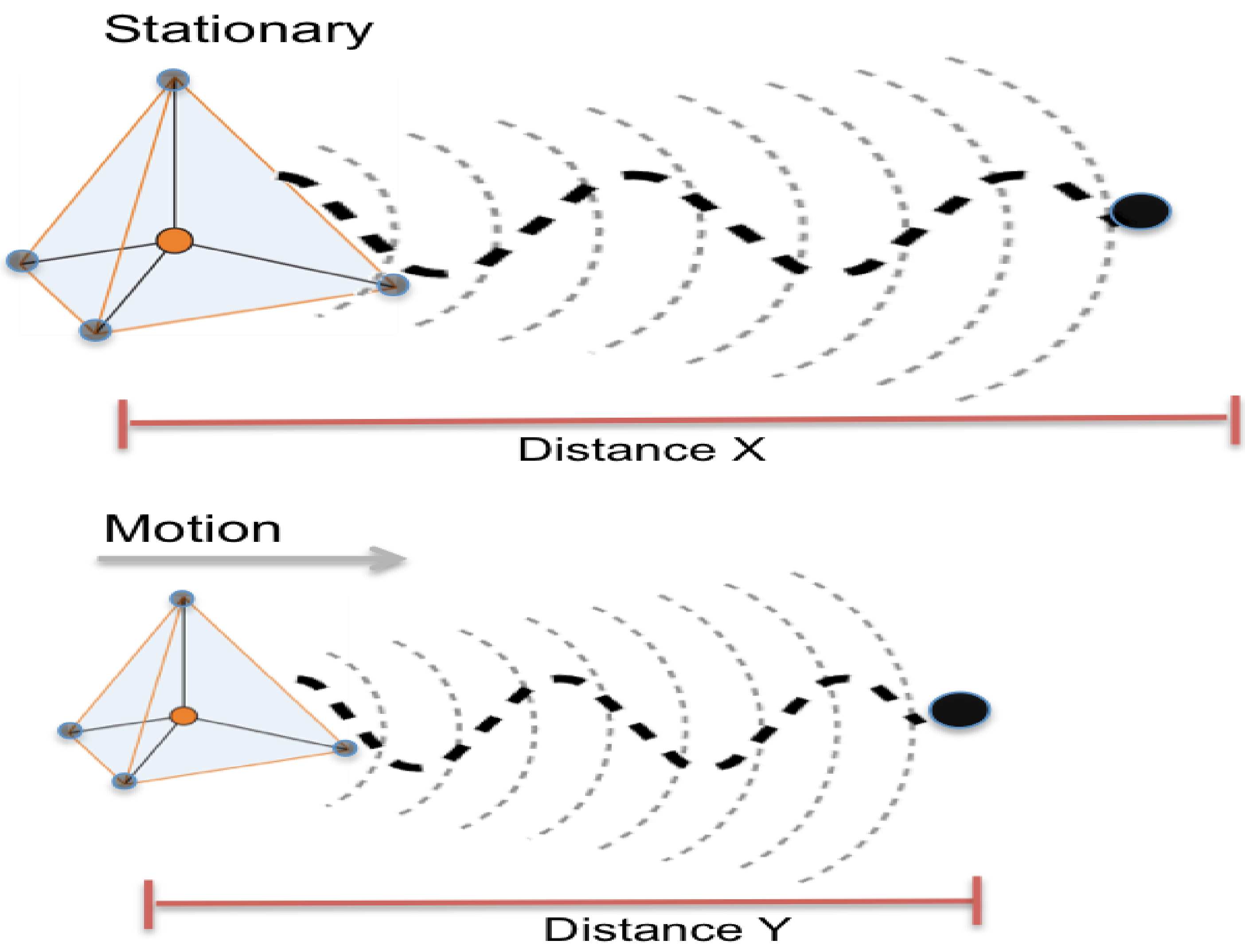
- The preceding results naturally seem a little strange to us.
- In relativity we often carry out **thought (or *gedanken* from the German word) experiments**.
- Consider the following *gedanken* experiment.
- **Mary**, in the moving system K′, flashes a light at her origin along her y' axis.
- The light travels a distance L, reflects off a mirror, and returns.
- **Mary** says that the total time for the journey is $T'_0 = t'_2 - t'_1 = 2L/c$, and this is indeed the proper time, because the clock in K beside Mary is at rest.





- Mary, in system K′, flashes a light along her y' axis and measures the proper time $T'_0 = 2L/c$ for the light to return.
- In system K, Frank will see the light travel partially down his x axis, because system K′ is moving.
- Fred times the arrival of the light in system K.
- The time interval T that Frank and Fred measure is related to the proper time by $T = \gamma T'_0$.

Length Contraction



Length Contraction

- Let's consider what might happen to the **length of objects in relativity**.
- Let an observer in each system K and K' **have a meterstick at rest** in his or her own respective system.
- Each observer lays the stick down along his or her respective x axis, putting the left end at x_l (or x_l') and the right end at x_r (or x'_r).
- **Frank in system K measures his stick to be $L_0 = x_r - x_l$.**
- Similarly, in system **K', Mary measures her stick at rest to be $L'_0 = x'_r - x_l' = L_0$.**
- Every observer measures a meterstick **at rest in his or her own system to have the same length**, namely 1 meter.
- The length as measured at rest is called the **proper length**.

- Let system K be at rest and system **K' move along the *x* axis** with speed *v*.
- Frank, who is at rest in system K, measures the length of the stick moving in K'.
- The difficulty is to measure the **ends of the stick simultaneously**.
- We insist that **Frank** measure the ends of the stick at the same time so that $t = t_r = t_l$.
- The events denoted by (x, t) are (x_l, t) and (x_r, t).
- We find

$$x'_r - x'_\ell = \frac{(x_r - x_\ell) - v(t_r - t_\ell)}{\sqrt{1 - v^2/c^2}}$$

Length Contraction

- The meterstick is at rest in system K', so the length $x_r' - x_l'$ must be the proper length L'_0.
- Denote the length measured by Frank as $L = x_r - x_l$.
- The times t_r and t_l are identical, as we insisted, so $t_r - t_l = 0$.
- Notice that the times of measurement by Mary in her system, t_l' and t'_r , are *not* identical.
- It makes no difference when Mary makes the measurements in her own system, because the stick is at rest.
- However, it makes a big difference when Frank makes his measurements, because the stick is moving with speed v with respect to him.
- **The measurements must be done simultaneously!**
- With these results, the previous equation becomes

$$L'_0 = \frac{L}{\sqrt{1 - v^2/c^2}} = \gamma L$$

or, because $L'_0 = L_0$,

$$L = L_0\sqrt{1 - v^2/c^2} = \frac{L_0}{\gamma}$$

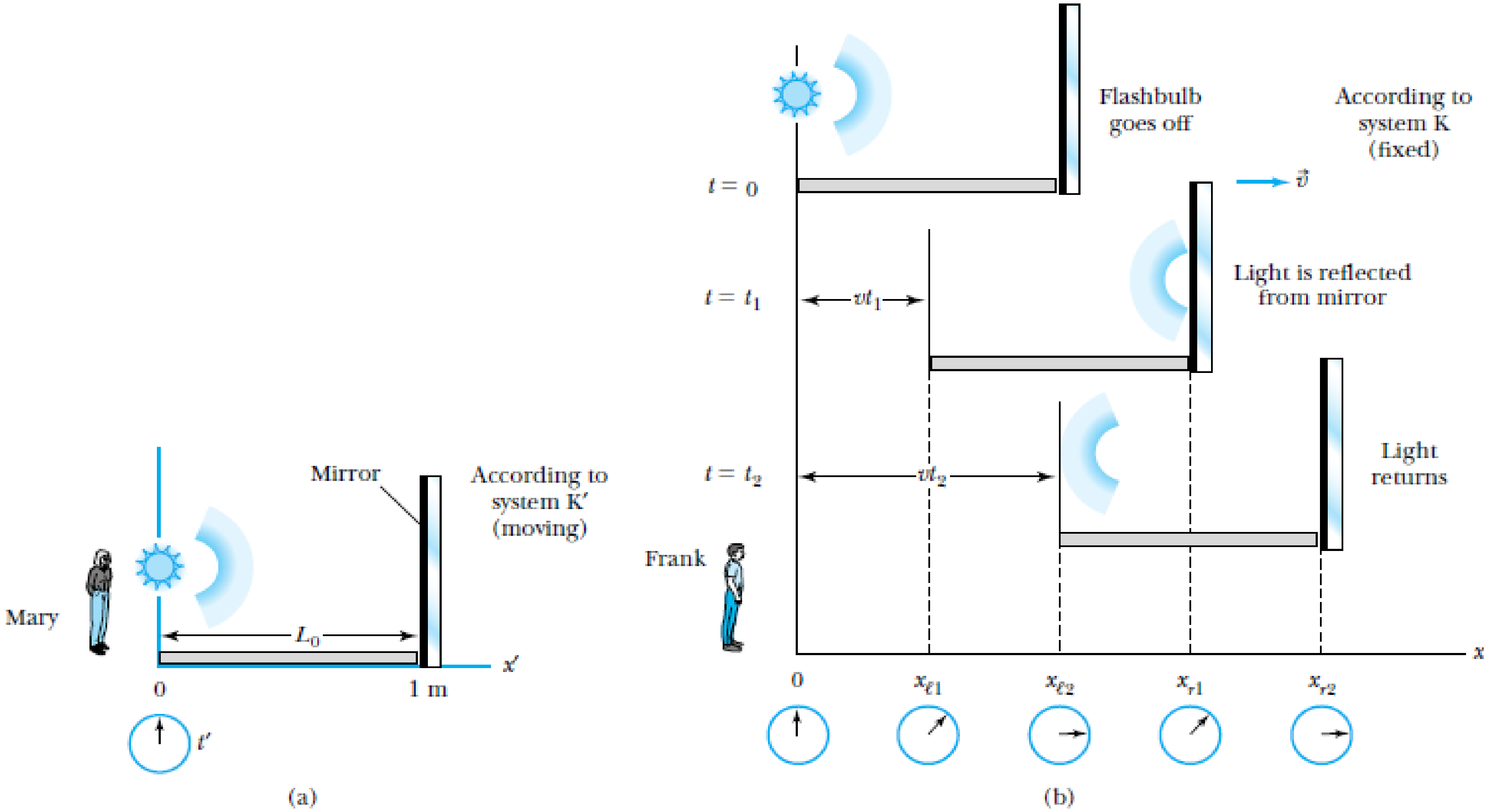
Length Contraction

- Notice that $L_0 > L$, so the moving meterstick **shrinks** according to Frank.
- This effect is known as length or space contraction and is characteristic of relative motion.
- This effect is also sometimes called the ***Lorentz-FitzGerald contraction*** because Lorentz and FitzGerald independently suggested the contraction as a way to solve the electrodynamics problem.
- This effect, like time dilation, is also reciprocal.
- **Each observer will say that the other moving stick is shorter.**
- There is no length contraction perpendicular to the relative motion, however, because $y' = y$ **and** $z' = z$.
- Observers in both systems can check the length of the other meterstick placed perpendicular to the direction of motion as the metersticks pass each other.
- They will agree that both metersticks are 1 meter long.

Length Contraction

- We can perform another ***gedanken* experiment** to arrive at the same result.
- This time we lay the meterstick along the x' axis in the moving system K' (Figure a).
- The two systems K and K' are **aligned** at $t = t' = 0$.
- A **mirror is placed** at the end of the meterstick, and a **flashbulb goes off** at the origin at $t = t' = 0$, sending a light pulse down the x' axis, where it is reflected and returned.
- **Mary** sees the stick at rest in system K' and measures the proper length L_0 (which should of course be one meter).
- **Mary** uses the same clock fixed at $x' = 0$ for the time measurements.
- The stick is moving **at speed *v* with respect to Frank** in the fixed system K.
- The clocks at $x = x' = 0$ both read zero when the origins are aligned just when the flashbulb goes off.
- Notice the situation shown in system K (Figure b), where **by the time the light reaches the mirror, the entire stick has moved a distance vt_1.**
- **By the time the light has been reflected back to the front of the stick again, the stick has moved a total distance vt_2.**

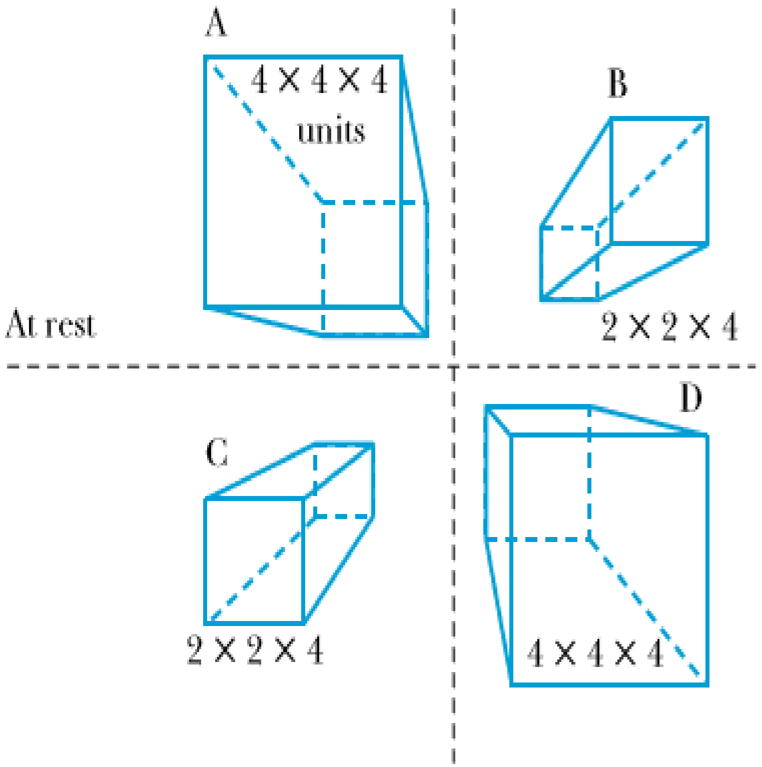
Length Contraction

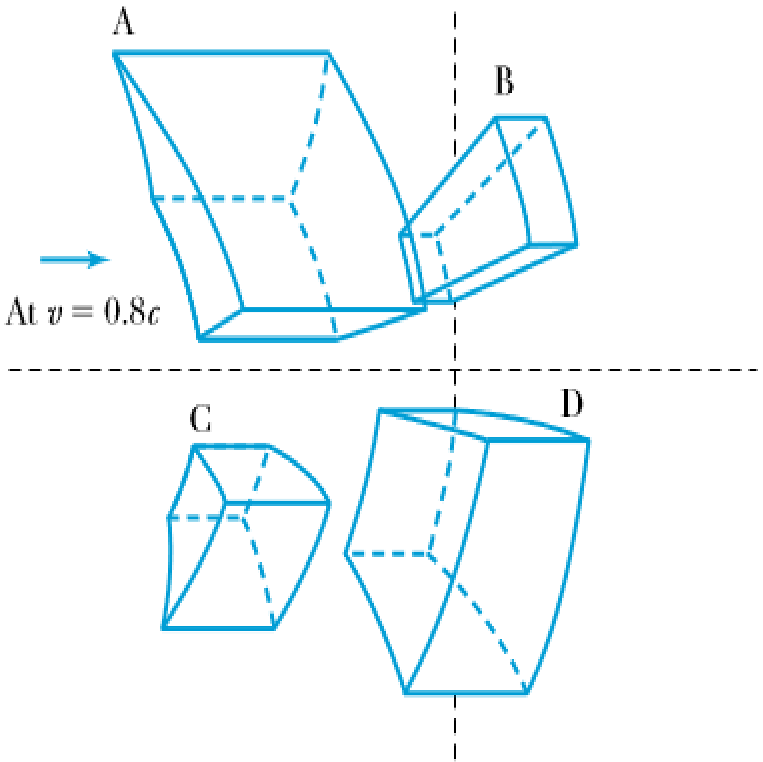


(a) Mary, in system K′, flashes a light down her x' axis along a stick at rest in her system of length L_0, which is the proper length. The time interval for the light to travel down the stick and back is $2L_0/c$. (b) Frank, in system K, sees the stick moving, and the mirror has moved a distance vt_1 by the time the light is reflected. By the time the light returns to the beginning of the stick, the stick has moved a total distance of vt_2. The times can be compared to show that the moving stick has been length contracted by $L = L_0\sqrt{1 - v^2/c^2}$.

Length Contraction

- The effect of length contraction along the direction of travel may strongly affect the appearances of two- and three-dimensional objects.
- **We see such objects when the light reaches our eyes, not when the light actually leaves the object**.
- **Thus, if the objects are moving rapidly, we will not see them as they appear at rest.**
- Figure shows the appearance of several such objects as they move.
- **Note that not only do the horizontal lines become contracted, but the vertical lines also become hyperbolas.**

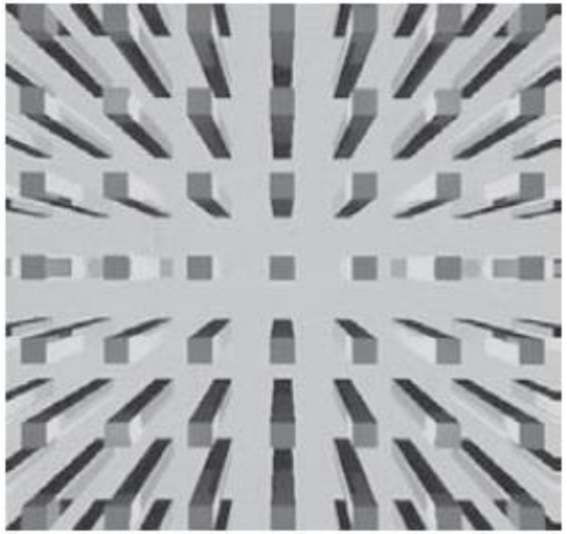




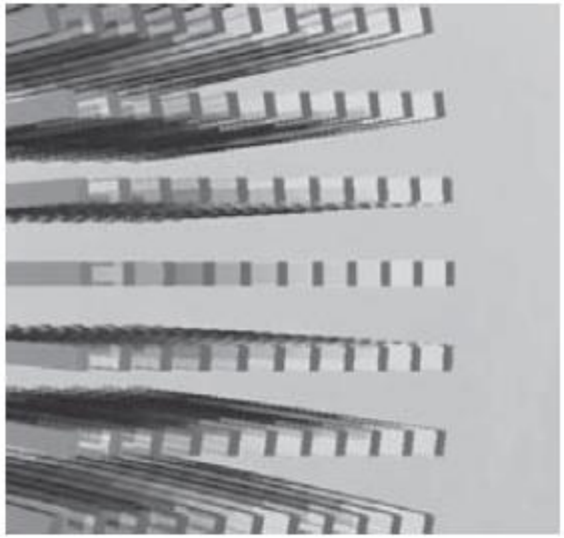
- In this **computer simulation**, the rectangular boxes are drawn as if the observer were 5 units in front of the near plane of the boxes and directly in front of the origin.
- The boxes are shown at rest on the left.
- On the right side, the boxes are moving to the right at a speed of v = 0.8c.
- **The horizontal lines are only length contracted, but notice that the vertical lines become hyperbolas.**
- The objects appear to be slightly rotated in space.
- The objects that are further away from the origin appear earlier because they are photographed at an earlier time and because the light takes longer to reach the camera (or our eyes).

- We show in Figure, a row of bars moving to the right with speed $v = 0.9c$. The result is quite surprising.

(a)



(b)



(a) An array of rectangular bars is seen from above at rest. (b) The bars are moving to the right at $v = 0.9c$. The bars appear to contract and rotate. *Quoted from P.-K. Hsuing and R. H. P. Dunn,* Science News ***137****,* *232 (1990).*