Special Theory of Relativity

LECTURE 4

Relativistic Momentum

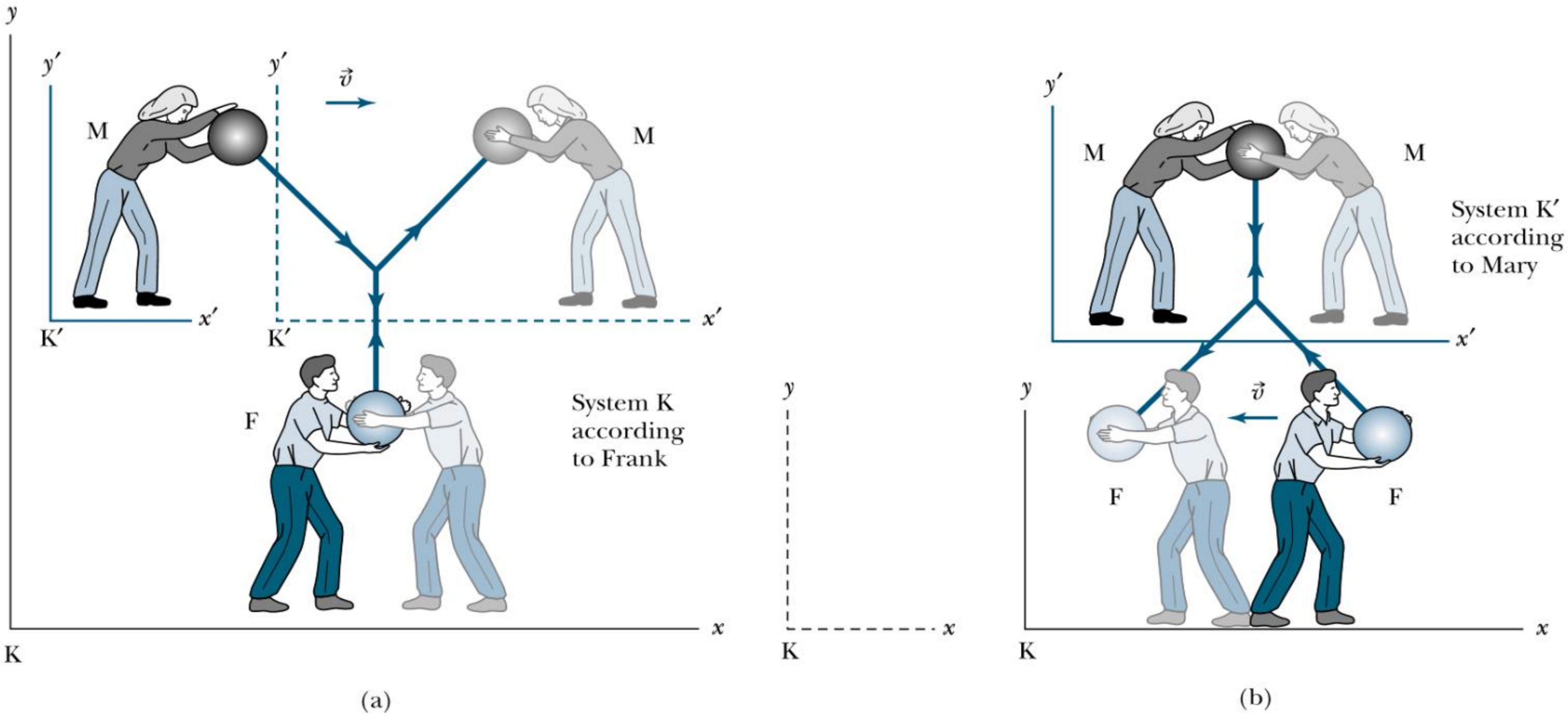
- Newton's second law, $\vec{F} = d\vec{p}/dt$, keeps its **same** form under a **Galilean transformation**, but we might **not** expect it to do so under a **Lorentz transformation**.
- There may be similar transformation difficulties with the conservation laws of **linear momentum and energy**.
- We need to look at our previous definition of linear momentum to see whether it is still **valid at high speeds**.
- According to **Newton's second law**, for example, an acceleration of a particle already moving at very high speeds could lead to a speed greater than the speed of light.
- That would be **in conflict with** the Lorentz transformation, so we expect that Newton's second law might somehow **be modified at high speeds**.

- Because **physicists believe the conservation of linear momentum** is fundamental, we begin by considering a collision that has no external forces.

- **Frank** (Fixed or stationary system) is at rest in system K holding a ball of mass **m**.

- **Mary** (Moving system) holds a similar ball in system K' that is moving in the **x direction** with velocity v with respect to system K.

- **Frank** throws his ball along his **y axis**, and Mary throws her ball with exactly the same speed along her **- y' axis**.



- Frank is in the fixed K system, and Mary is in the moving K’ system. Frank throws his ball along his **+y axis**, and Mary throws her ball along her **–y’ axis**. The balls collide. **The event is shown in Frank’s system in (a) and in Mary’s system in (b).** (Because it is awkward to show the twins as they catch the ball, we have drawn them faintly and in a reversed position.)

- The two balls **collide** in a perfectly **elastic collision**, and each of them catches their own ball as it rebounds.

- Each twin measures the speed of his or her own ball to be $\boldsymbol{u_0}$ both **before and after the collision**.

- We show the collision according to both observers in Figure.

- **Consider the conservation of momentum according to Frank as seen in system K.**

- The **velocity of the ball** thrown by Frank has components in his own system K of

$$u_{Fx} = 0$$

$$u_{Fy} = u_0$$

- If we use the definition of momentum, $\vec{p} = m\vec{v}$**,** the momentum of the ball thrown by Frank is entirely in the y direction:

$$p_{Fy} = mu_0$$

- Because the **collision is perfectly elastic**, the ball returns to Frank with speed $\boldsymbol{u_0}$ **along the -y axis**.
- The change of momentum of his ball as observed by Frank in system K is

$$\Delta p_F = \Delta p_{Fy} = -2mu_0$$

- In order to confirm the conservation of linear momentum, we need to **determine the change in the momentum of Mary's ball** as measured by Frank.
- We will let the primed speeds be measured by Mary and the unprimed speeds be measured by Frank (except that u_0 is always the speed of the ball as measured by the twin in his or her own system).
- Mary measures the initial velocity of her own ball to be $u'_{Mx} = 0$ and **$u'_{My} = -\,u_0$, because she throws it along her own –y' axis**.

- To determine the velocity of **Mary's ball as measured by Frank**, we need to use the velocity transformation equations.
- If we insert the appropriate values for the speeds just discussed, we obtain

$$u_{Mx} = v$$
$$u_{My} = -u_0\sqrt{1 - v^2/c^2}$$

- Before the collision, the momentum of **Mary's ball as measured by Frank** becomes

$$\text{Before} \quad p_{Mx} = mv$$
$$\text{Before} \quad p_{My} = -mu_0\sqrt{1 - v^2/c^2}$$

- For a perfectly elastic collision, the momentum after the collision is

$$\text{After} \quad p_{Mx} = mv$$

$$\text{After} \quad p_{My} = +mu_0\sqrt{1 - v^2/c^2}$$

- The change in momentum of **Mary's ball according to Frank** is

$$\Delta p_M = \Delta p_{My} = 2mu_0\sqrt{1 - v^2/c^2}$$

- The **conservation of linear momentum** requires the total change in momentum of the collision, $\mathbf{\Delta p_F + \Delta p_M}$, to be zero.
- When we look at these equations we can see that **does not give zero**.

- *Linear momentum is not conserved if we use the conventions for momentum from classical physics even if we use the velocity transformation equations from the special theory of relativity.*
- There is no problem with **the x direction**, but there is a problem with the **y direction** along the direction the ball is thrown in each system.
- Rather than abandon the conservation of linear momentum, let us look for a **modification of the definition of linear momentum** that preserves both it and Newton's second law.
- We follow a procedure similar to the one we used in deriving the Lorentz transformation; **we assume the simplest, most reasonable change that may preserve the conservation of momentum.**
- We assume that the classical form of momentum $m\vec{u}$ is multiplied by a factor that may depend on velocity.

- Let the factor be $\Gamma(\boldsymbol{u})$.
- Our trial definition for linear momentum now becomes

$$\vec{p} = \Gamma(u) m\vec{u}$$

- **Momentum is conserved in the collision** just described for the value of $\Gamma(\boldsymbol{u})$ given by

$$\Gamma(u) = \frac{1}{\sqrt{1 - u^2/c^2}}$$

- Notice that the form of Equation is the same as that found earlier for the Lorentz transformation.
- $\boldsymbol{\Gamma}$**(u)** $= \boldsymbol{\gamma}$
- However, **this** γ **is different**; **it contains the speed of the particle u**, whereas the Lorentz transformation contains the relative speed v between the two inertial reference frames.

- We can make a plausible determination for the correct form of the momentum **if we use the proper time** discussed previously to determine the velocity.
- The momentum becomes

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = m\frac{d\vec{r}}{dt}\frac{dt}{d\tau}$$

- We retain the velocity $\vec{\boldsymbol{u}} = {}^{\boldsymbol{d\vec{r}}}/_{\boldsymbol{dt}}$ as used classically, where $\vec{r}$ is the position vector.
- All observers **do not agree** as to the value of ${}^{d\vec{r}}/_{dt}$, but they do agree as to the value of ${}^{d\vec{r}}/_{d\tau}$, where $d\tau$ is the **proper time measured in the moving system K'**.

- The value of **dt / $d\tau$ (= γ)** is obtained, where using the speed u in the relation for γ to represent the relative speed of the moving (Mary's) frame and the fixed (Frank's) frame.
- The definition of the **relativistic momentum** becomes,

$$\vec{p} = m\frac{d\vec{r}}{dt}\gamma$$

$$\vec{p} = \gamma m\vec{u} \quad \text{Relativistic momentum}$$

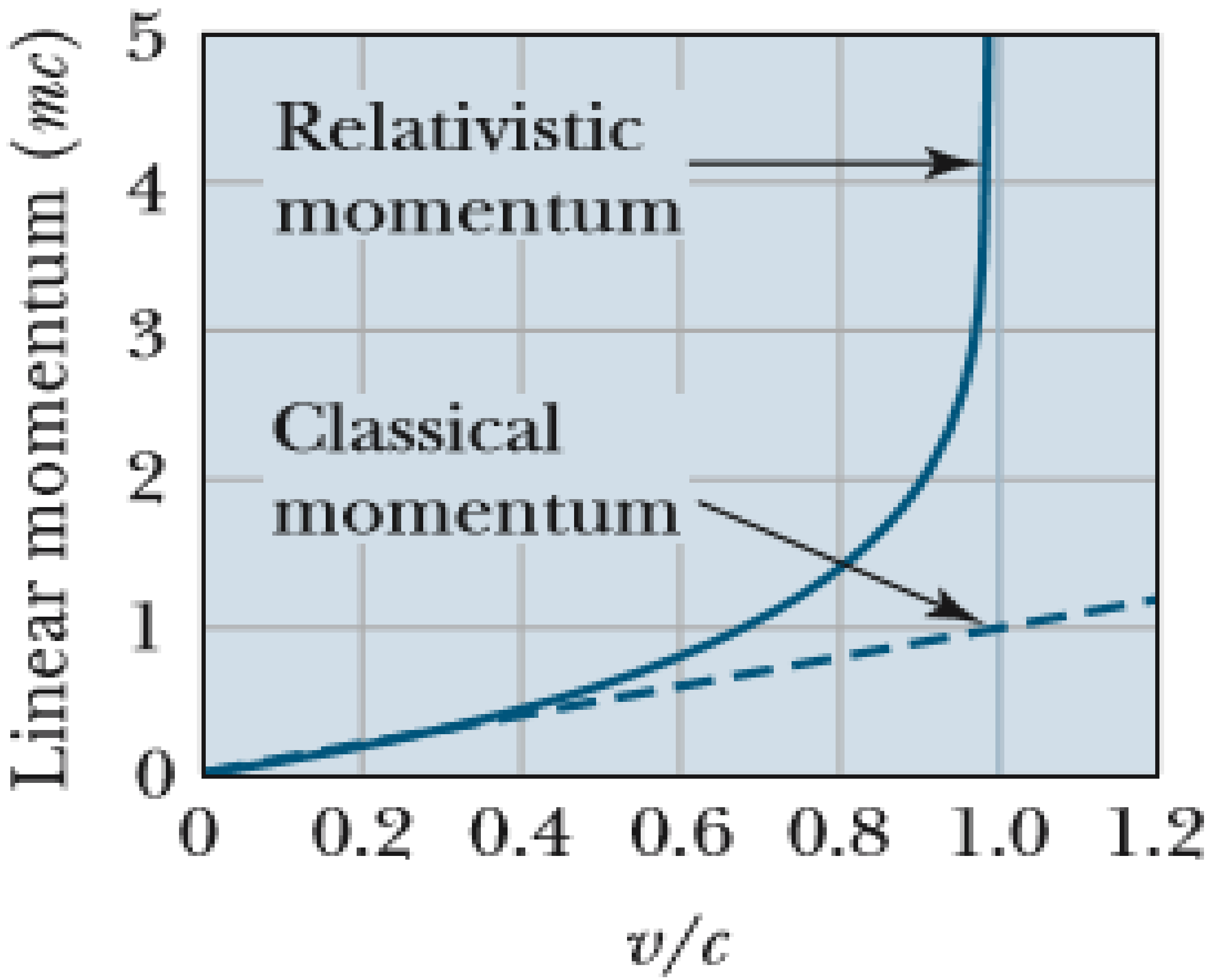
where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

- **This result for the relativistic momentum reduces to the classical result for small values of $\boldsymbol{u/c}$.**

- The classical momentum expression is good to an accuracy of 1% as long as $u < 0.14c$.

- We show both the **relativistic and classical** momentum in Figure.



The linear momentum of a particle of mass m is plotted versus its velocity (v/c) for both the classical and relativistic momentum results. As $v \to c$ the relativistic momentum becomes quite large, but the classical momentum continues its linear rise. The relativistic result is the correct one.

Relativistic Energy

- The concept of force is best defined by its use in Newton’s laws of motion, and we retain here the classical definition of force as used in Newton’s second law.
- We studied the concept of momentum and found a relativistic expression.
- Therefore, we modify Newton’s second law to include our new definition of linear momentum, and force becomes

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{u}) = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}\right)$$

- **Introductory physics presents kinetic energy as the work done on a particle by a net force.**
- We retain here the same definitions of kinetic energy and work.
- The work $\boldsymbol{W_{12}}$ done by a force $\vec{\boldsymbol{F}}$ to move a particle from position 1 to position 2 along a path $\vec{\boldsymbol{s}}$ is defined to be

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} = K_2 - K_1$$

- where K_1 is defined to be the kinetic energy of the particle at position 1.

- For simplicity, let the particle start from rest under the influence of the force $\vec{\boldsymbol{F}}$ and calculate the final kinetic energy K after the work is done.
- The force is related to the dynamic quantities.
- The work W and kinetic energy K are

$$W = K = \int \frac{d}{dt}(\gamma m\vec{u}) \cdot \vec{u}\, dt$$

where the integral is performed over the differential path $\mathbf{d}\vec{\boldsymbol{s}} = \vec{\boldsymbol{u}}\boldsymbol{dt}$.

- Because the **mass is invariant**, it can be brought outside the integral.
- The **relativistic factor** γ **depends on** $\boldsymbol{u}$ and cannot be brought outside the integral.

- Equation becomes

$$K = m\int dt\,\frac{d}{dt}(\gamma\vec{u})\cdot\vec{u} = m\int u\,d(\gamma u)$$

- The limits of integration are from an initial value of **0** to a final value of γ**u**.

$$K = m\int_0^{\gamma u} u\,d(\gamma u)$$

- The integral is straightforward if done by the method of integration by parts.
- The result, called the **relativistic kinetic energy**, is

$$K = \gamma mc^2 - mc^2 = mc^2\left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1\right) = mc^2(\gamma - 1)$$

- This does not seem to resemble the classical result for kinetic energy, $\boldsymbol{K = \frac{1}{2}mu^2}$.
- However, if it is correct, we expect it to reduce to the classical result for low speeds.
- **This equation is particularly useful when dealing with particles accelerated to high speeds.**

- For speeds $\boldsymbol{u} \ll \boldsymbol{c}$, we expand γ in a binomial series as follows:

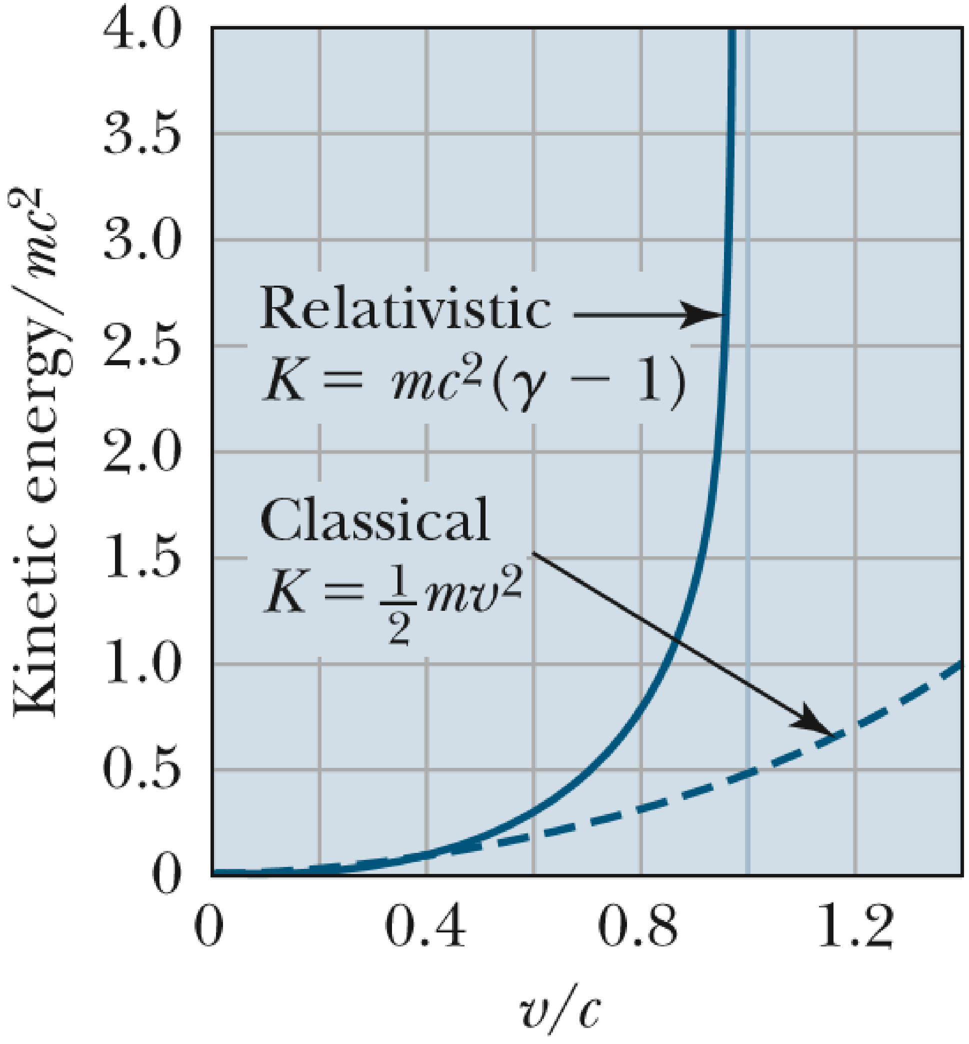
$$K = mc^2\left(1 - \frac{u^2}{c^2}\right)^{-1/2} - mc^2$$
$$= mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2$$

- where we have neglected all terms of power $(\boldsymbol{u}/\boldsymbol{c})^{\boldsymbol{4}}$ and greater, because $u \ll c$.
- This gives the following equation for the **relativistic kinetic energy at low speeds**:

$$K = mc^2 + \frac{1}{2}mu^2 - mc^2 = \frac{1}{2}mu^2$$

- which is the **expected classical result**.

- We show both the relativistic and classical kinetic energies in Figure.
- They diverge considerably **above a velocity of 0.6c**.



The kinetic energy as a fraction of rest energy (K/mc^2) of a particle of mass m is shown versus its velocity (v/c) for both the classical and relativistic calculations. Only the relativistic result is correct. Like the momentum, the kinetic energy rises rapidly as $v \rightarrow c$.

Total Energy and Rest Energy

- We rewrite

$$K = \gamma mc^2 - mc^2 = mc^2\left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1\right) = mc^2(\gamma - 1)$$

this equation in the form

$$\gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = K + mc^2$$

mc^2 is called the **rest energy** and is denoted by E_0.

$$E_0 = mc^2$$

- This leaves the sum of the kinetic energy and rest energy to be interpreted as the **total energy** of the particle.

- The total energy is denoted by E and is given by

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \frac{E_0}{\sqrt{1 - u^2/c^2}} = K + E_0$$

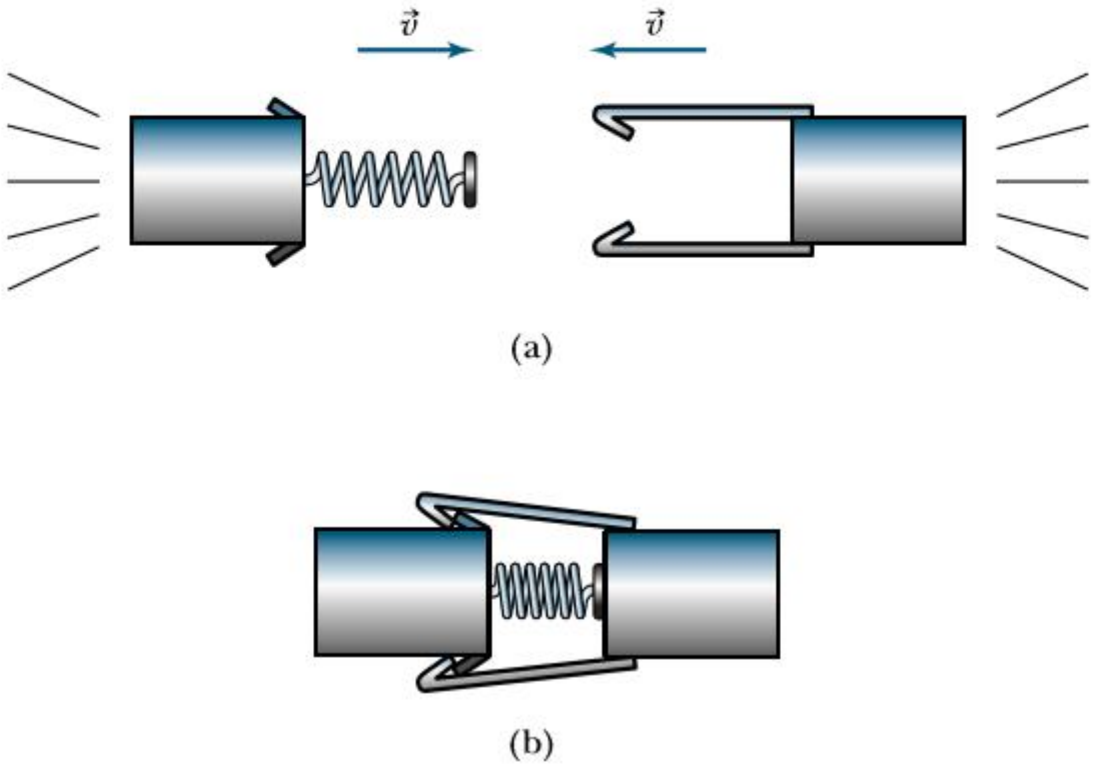
Equivalence of Mass and Energy

- These last few equations suggest the **equivalence of mass and energy**, a concept attributed to Einstein.
- The result that **energy = mc^2** is one of the most famous equations in physics.
- Even when a particle has no velocity, and thus no kinetic energy, we still believe that the particle has energy through its mass, **$E_0 = mc^2$**.
- Nuclear reactions are certain proof that **mass and energy are equivalent**.
- The concept of motion as being described by **kinetic energy** is preserved in relativistic dynamics, **but a particle with no motion still has energy through its mass.**

- In order to establish the equivalence of mass and energy, we must **modify two of the conservation laws** that we learned in classical physics.
- Mass and energy are **no longer two separately conserved** quantities.
- We must combine them into one law of the **conservation of mass-energy.**
- We will see ample proof during the remainder of this book of the validity of this basic conservation law.

- Even though we often say "**energy is turned into mass**" or "**mass is converted into energy**" or "**mass and energy are interchangeable**," what we mean is that mass and energy are equivalent.
- **Mass is another form of energy**, and we use the terms mass-energy and energy interchangeably.
- This is not the first time we have had to change our understanding of energy.
- In the late eighteenth century it became clear that heat was another form of energy, and the nineteenth-century experiments of James Joule showed that heat loss or gain was related to work.

- Consider two blocks of wood, each of **mass m** and having **kinetic energy K**, moving toward each other as shown in Figure.
- A **spring** placed between them is compressed and locks in place as they collide.



(a) Two blocks of wood, one with a spring attached and both having mass m, move with equal speeds v and kinetic energies K toward a head-on collision. (b) The two blocks collide, compressing the spring, which locks in place. The system now has increased mass, $M = 2m + 2K/c^2$, with the kinetic energy being converted into the potential energy of the spring.

- Let’s examine the **conservation of mass-energy**.
- The **energy before the collision is**

$$\text{Mass-energy before:} \qquad E = 2mc^2 + 2K$$

and the **energy after the collision** is

$$\text{Mass-energy after:} \qquad E = Mc^2$$

- **M is the rest mass of the system**.
- Because energy is conserved, we have

$$E = 2mc^2 + 2K = Mc^2$$

and the new **mass M** is greater than the individual masses **2m**.

- The kinetic energy went into compressing the spring, so the spring has increased **potential energy**.

- Kinetic energy has been converted into mass, the result being that the potential energy of the spring has caused the system to have more mass.
- We find the **difference in mass ΔM** by setting the previous two equations for energy equal and solving for **ΔM = M - 2m.**

$$\Delta M = M - 2m = \frac{2K}{c^2}$$

- Linear momentum is conserved in this head-on collision.
- The fractional mass increase in this case is quite small and is given by **f_r = ΔM/2m.**

- So we have

$$f_r = \frac{M - 2m}{2m} = \frac{2K/c^2}{2m} = \frac{K}{mc^2}$$

- For typical masses and kinetic energies of blocks of wood, this fractional increase in mass **is too small to measure**.
- For example, if we have blocks of wood of mass 0.1 kg moving at 10 m/s,

$$\boldsymbol{f_r = \frac{\frac{1}{2}mv^2}{mc^2} = \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}\frac{(10\ m/s)^2}{(3x10^8\ m/s)^2} = 6x10^{-16}}$$

- In that equation we have used the **nonrelativistic expression** for kinetic energy because the **speed is so low**.

- This very small numerical result indicates that questions of mass increase are inappropriate for macroscopic objects such as blocks of wood and automobiles crashing into one another.

- Such small increases cannot now be measured, but we will look at the collision of two high-energy protons, in which considerable energy is available to create additional mass.

- **Mass-energy relations are essential in such reactions**.

Relationship of Energy and Momentum

- Physicists believe that **linear momentum is a more fundamental concept than kinetic energy**.

- There is **no conservation of kinetic energy**, whereas the conservation of linear momentum in isolated systems is inviolate as far as we know.

- We begin with this equation for the relativistic momentum written in magnitude form only.

$$p = \gamma m u = \frac{mu}{\sqrt{1 - u^2/c^2}}$$

- We square this result, multiply by c^2, and rearrange the result

$$\begin{aligned} p^2c^2 &= \gamma^2 m^2 u^2 c^2 \\ &= \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) = \gamma^2 m^2 c^4 \beta^2 \end{aligned}$$

- We use for $\boldsymbol{\beta}^2 = \frac{\boldsymbol{\gamma}^2 - \mathbf{1}}{\boldsymbol{\gamma}^2}$ and find

$$p^2c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$$

$$= \gamma^2 m^2 c^4 - m^2 c^4$$

- The first term on the right-hand side is just $\boldsymbol{E^2}$, and the second term is $\boldsymbol{E_0^2}$. The last equation becomes

$$p^2c^2 = E^2 - E_0^2$$

- We rearrange this last equation to find the result we are seeking, a relation between energy and momentum.

$$E^2 = p^2c^2 + E_0^2$$

Momentum- energy relation

- or

$$E^2 = p^2c^2 + m^2c^4$$

- **Momentum- energy relation** is a useful result to relate the total energy of a particle with its momentum.

- **The quantities $(\boldsymbol{E^2 - p^2c^2})$ and *m* are invariant quantities**.

- Note that when a particle's velocity is zero and it has no momentum, **Momentum-energy relation** correctly gives $\boldsymbol{E_0}$ as the particle's total energy.

Massless Particles

- This equation $E^2 = p^2c^2 + E_0^2$ can also be used to determine the total energy for particles having zero mass.

- For example, this equation predicts that the total energy of a photon is

$$E = pc \qquad Photon$$

- **The energy of a photon is completely due to its motion**.

- **It has no rest energy, because it has no mass**.

- We can show that the previous relativistic equations correctly predict that the speed of a photon must be the speed of light *c*.

- For the total energy of a photon we set these two equations equal.

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \frac{E_0}{\sqrt{1 - u^2/c^2}} = K + E_0$$

and

$$E = pc \qquad Photon$$

and get

$$E = \gamma mc^2 = pc$$

- If we insert the value of the relativistic momentum from Relativistic momentum equato), we have

$$\gamma mc^2 = \gamma muc$$

- The fact that $u = c$ follows directly from this equation after careful consideration of letting $m \rightarrow 0$ and realizing that $\gamma \rightarrow \infty$.

$$u = c \qquad Massless\ particle$$

- **Massless particles must travel at the speed of light.**