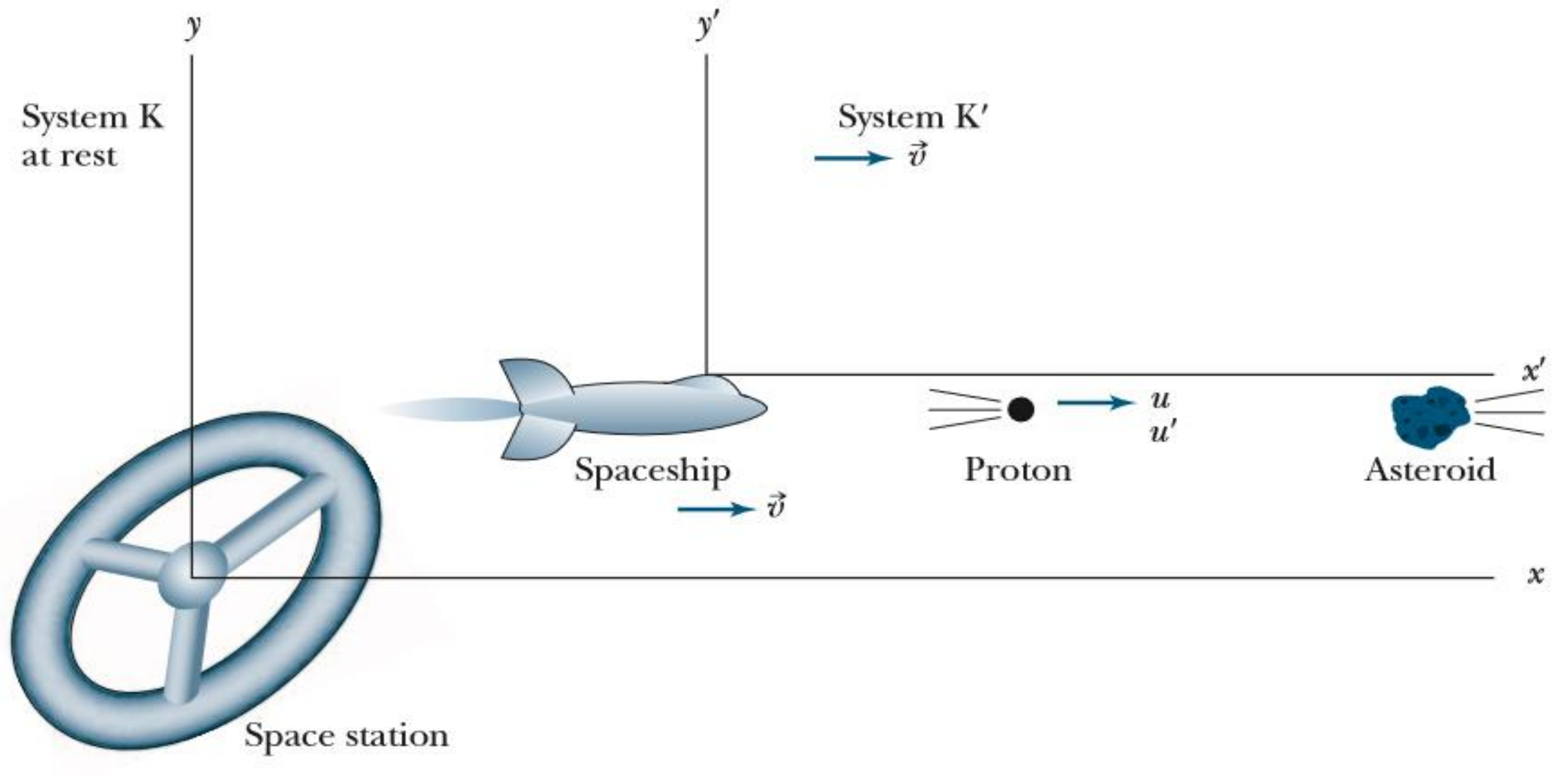
Special Theory of Relativity

LECTURE 3

Addition of Velocities

- A spaceship launched from a space station quickly reaches its cruising speed of $\mathbf{0.60}\boldsymbol{c}$ with respect to the space station when a band of asteroids is observed straight ahead of the ship.



- **Mary**, the commander, reacts quickly and orders her crew to blast away the asteroids with the ship’s proton gun to avoid a catastrophic collision.

- **Frank**, the admiral on the space station, listens with apprehension to the communications because he fears the asteroids may eventually destroy his space station as well.

- Will the high-energy protons of speed $\mathbf{0.99}\boldsymbol{c}$ be able to successfully blast away the asteroids and save both the spaceship and space station?

- If $\mathbf{0.99}\boldsymbol{c}$ is the speed of the protons with respect to the spaceship, what speed will Frank measure for the protons?

- **Frank** (in the fixed, stationary system K on the space station) will measure the velocity of the protons to be $\boldsymbol{u}$, whereas **Mary**, the commander of the spaceship (the moving system K'), will measure
  u' = 0.99c.
- The velocity of the spaceship with respect to the space station is
  $v =$ 0.60c.
- **Newtonian mechanics** teaches us that to find the velocity of the protons with respect to the space station, we simply add the velocity of the spaceship with respect to the space station (0.60c) to the velocity of the protons with respect to the spaceship (0.99c) to determine the result $\underline{\boldsymbol{u = v + u' = 0.60\, c + 0.99\, c = 1.59\, c}}$
- However, this result is **not in agreement** with the results of the Lorentz transformation.

- Taking the differentials of x :

$$dx = \gamma(dx' + v\,dt')$$

$$dy = dy'$$

$$dz = dz'$$

$$dt = \gamma[dt' + (v/c^2)\,dx']$$

- Velocities are defined by

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma[dt' + (v/c^2)\,dx']} = \frac{u'_x + v}{1 + (v/c^2)u'_x}$$

Similarly, u_y and u_z are determined to be

$$u_y = \frac{u'_y}{\gamma[1 + (v/c^2)u'_x]}$$

$$u_z = \frac{u'_z}{\gamma[1 + (v/c^2)u'_x]}$$

These equations are referred to as the **Lorentz velocity transformations.**

Addition of Velocities

- Although the relative motion of the systems K and K’ is only along the ***x* direction**, the velocities along ***y* and *z*** are affected as well.
- This contrasts with the Lorentz transformation equations, where *y* = *y*’ and *z* = *z*’.
- The **difference in velocities** is simply ascribed to the transformation of time, which depends on v and x'.
- Thus, the transformations for u_y and u_z depend on v and u'_x.
- The **inverse transformations** for u'_x , u'_y, and u'_z can be determined by simply switching primed and unprimed variables and changing $\boldsymbol{v}$ **to** $-\boldsymbol{v}$.

$$u'_x = \frac{u_x - v}{1 - (v/c^2)u_x}$$

$$u'_y = \frac{u_y}{\gamma[1 - (v/c^2)u_x]}$$

$$u'_z = \frac{u_z}{\gamma[1 - (v/c^2)u_x]}$$

Addition of Velocities

- What is the **correct result for the speed of the protons** with respect to the space station?
- We have $u'_x = 0.99c$ and $v = 0.60c$, so this gives us the result

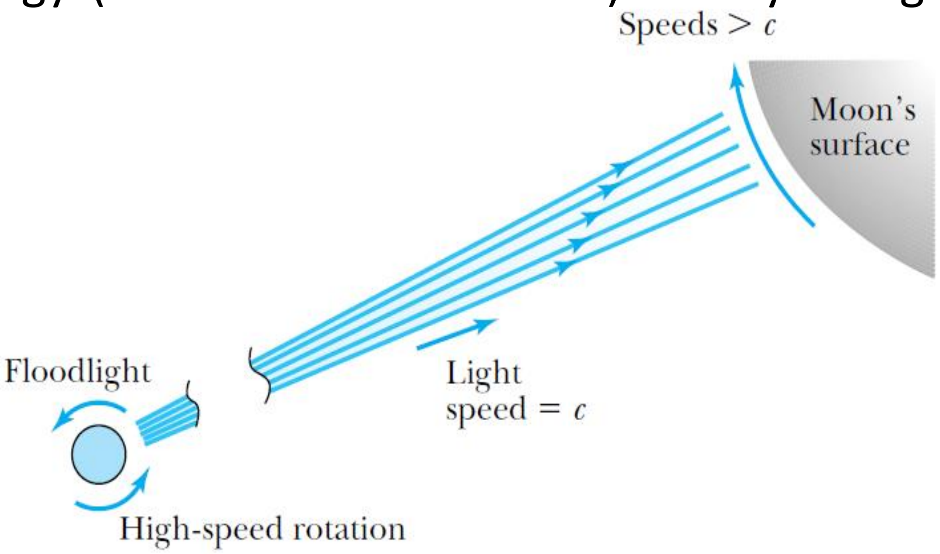
$$u_x = \frac{0.990c + 0.600c}{1 + \frac{(0.600c)(0.990c)}{c^2}} = 0.997c$$

where we have assumed we know the speeds to three significant figures.

- Therefore, **the result is a speed only slightly less than *c***.
- The Lorentz transformation does not allow a material object to have a speed greater than *c*.
- **Only massless particles, such as light, can have speed *c***.
- If the crew members of the spaceship spot the asteroids far enough in advance, their reaction times should allow them to shoot down the uncharacteristically swiftly moving asteroids and **save both the spaceship and the space station.**

Addition of Velocities

- Although **<u>no particle with mass can carry energy faster than *c*</u>**, we can imagine a signal being processed faster than *c*.
- Consider the following ***gedanken*** experiment.
- A giant floodlight placed on a space station above the Earth revolves at 100 Hz.
- Light spreads out in the **radial direction** from the floodlight at speeds of *c.*
- On the surface of the moon, the light beam sweeps across at speeds far exceeding *c*.
- **However, the light itself does not reach the moon at speeds faster than *c*.**
- No energy is associated with the beam of light sweeping across the moon's surface.
- The energy (and linear momentum) is only along the radial direction from the space station to the moon.



A floodlight revolving at high speeds can **sweep a light beam across the surface of the moon at speeds exceeding c,** but the **speed of the light still does not exceed c**.

Experimental Verification

- We have used the special theory of relativity to describe some unusual phenomena.

- The special theory has also been used to make some predictions concerning **length contraction, time dilation, and velocity addition**.

- Now we will discuss only **a few of the many experiments** that have been done to **confirm** the **special theory of relativity**.

Muon Decay

- When high-energy particles called ***cosmic rays*** enter the Earth's atmosphere from outer space, they interact with particles in the upper atmosphere, creating additional particles in a ***cosmic shower.***
- Many of the particles in the shower are p-*mesons* (pions), which **decay** into other **unstable** particles called ***muons***.
- Because muons are **unstable**, they decay according to the radioactive decay law

$$N = N_0 \exp\left(-\frac{(\ln 2)t}{t_{1/2}}\right) = N_0 \exp\left(-\frac{0.693t}{t_{1/2}}\right)$$

where N_0 and N are the number of muons at times $t = 0 \; and \; t = t,$ respectively, and $t_{1/2}$ is the **half-life of the muons**.

Muon Decay

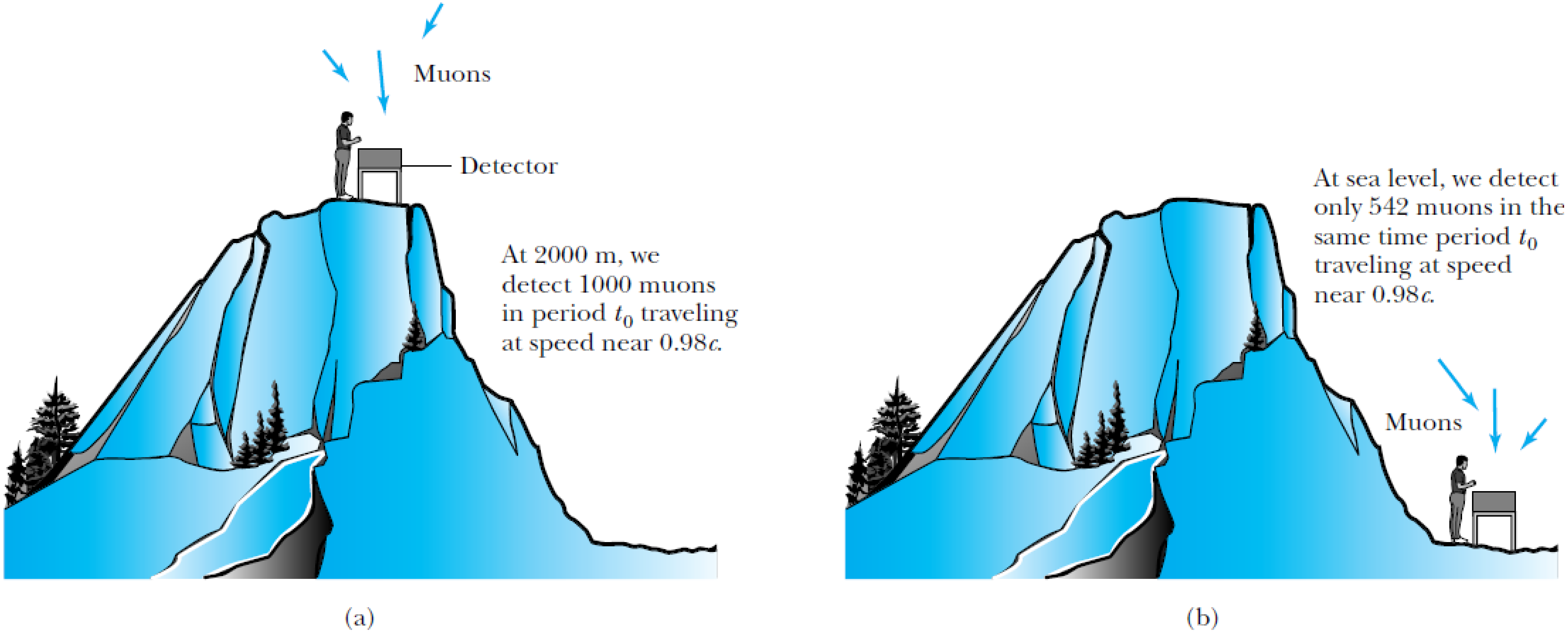
- This means that in the time period $t_{1/2}$ half of the muons will **decay to other particles**.
- The half-life of muons ($1.52\text{x}10^{6}$ s) is **long enough** that many of them survive the trip through the atmosphere to the **Earth's surface**.
- Much of what we know about muons in cosmic rays was learned from **balloon flights carrying sophisticated detectors**.
- This balloon is being prepared for launch in NASA's Ultra Long Duration Balloon program for a mission that may last up to 100 days.
- The payload will hang many meters below the balloon.
- Victor Hess began the first such balloon flights in 1912 (when he discovered cosmic rays), and much improved versions are still launched today from all over the world to study **cosmic rays, the atmosphere, the sun, and the universe**.

Photo courtesy of NASA.

Muon Decay

- We perform an experiment by placing a **muon detector** on top of a mountain **2000 m** high and counting the number of muons traveling at a speed near $\boldsymbol{v = 0.98c}$.
- Suppose we count **10^3 muons** during a given time period $\boldsymbol{t_0}$.
- We then move our muon detector to sea level, and we determine experimentally that approximately **540 muons** survive the trip without decaying.
- **Classically,** muons traveling at a speed of $\boldsymbol{0.98c}$ cover the 2000-m path in **$6.8\text{x}10^6$ s**, and according to the radioactive decay law, only **45 muons should** survive the trip.
- There is obviously **something wrong with the classical calculation**, because we counted **a factor of 12 more muons** surviving than the classical calculation predicts.

Muon Decay



The number of muons detected with speeds near $0.98c$ is much different (a) on top of a mountain than (b) at sea level, because of the muon's decay. The experimental result agrees with our time dilation equation.

Muon Decay

- Because the classical calculation **does not agree with the experimental result**, we should consider a **relativistic calculation**.

- The muons are moving at a speed of $\mathbf{0.98}\boldsymbol{c}$ with respect to us on Earth, so the effects of **time dilation** will be dramatic.

- In the muon rest frame, the time period for the muons to travel 2000 m (on a clock fixed with respect to the mountain) is calculated to be $(\mathbf{6.8/5.0})\ \boldsymbol{x}\ \mathbf{10^6}\ \boldsymbol{s}$, because $\boldsymbol{\gamma} = \mathbf{5.0}$ for $\boldsymbol{v} = \mathbf{0.98}\boldsymbol{c}$.

- For the time $\boldsymbol{t} = \mathbf{1.36}\ \boldsymbol{x}\ \mathbf{10^6}\ \boldsymbol{s}$, the radioactive decay law predicts that **538 muons** will survive the trip, in agreement with the observations.

Muon Decay

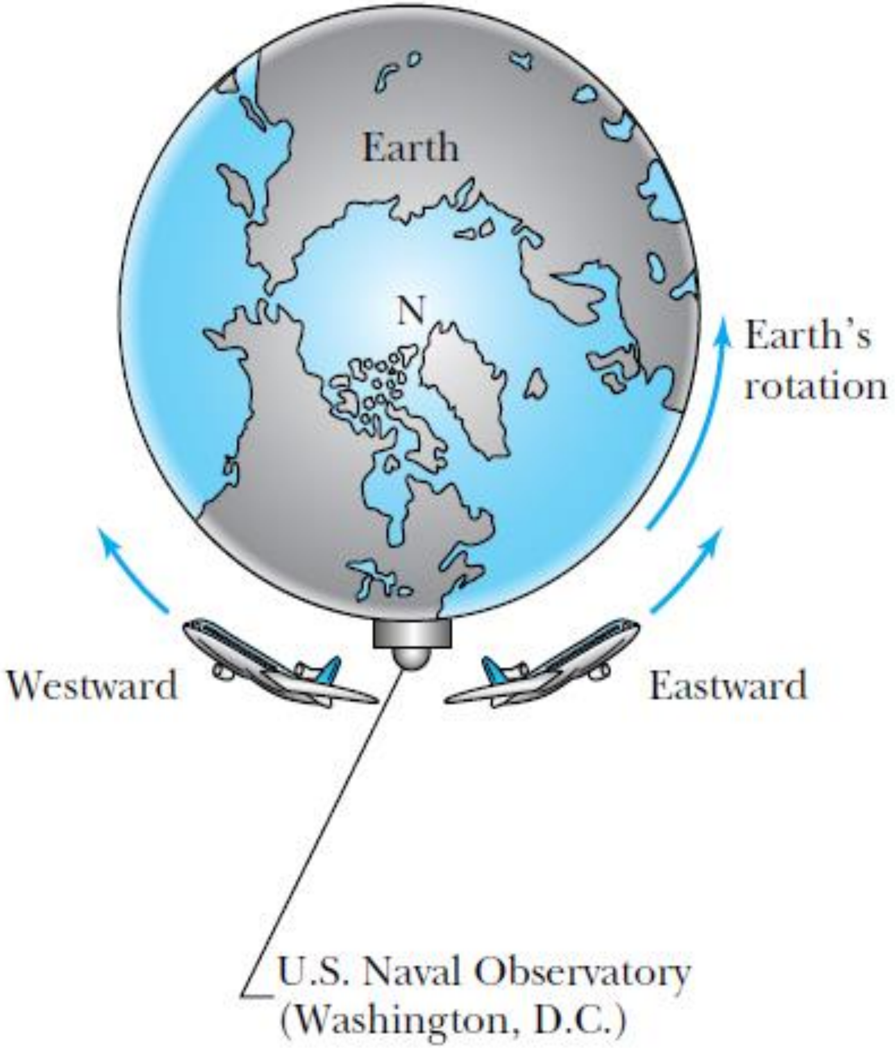
- It is useful to examine the muon decay problem from the perspective of an **observer traveling with the muon**.
- This observer **would not measure** the distance from the top of the mountain to sea level to be 2000 m.
- Rather, this observer would say that the **distance is contracted** and is only $(2000\,m)/5.0 \;=\; \boldsymbol{400\ m}$.
- The time to travel the 400-m distance would be $\frac{400\ m}{0.98c} = 1.36\ x\ 106\ s$ according to a **clock at rest with a muon**.
- Using the **radioactive decay law**, an observer traveling with the muons would still predict **538 muons** to survive.
- Therefore, we obtain the identical result whether we consider **time dilation or space contraction**, and both are in **agreement with the experiment**, thus confirming the special theory of relativity.

Atomic Clock Measurement

- In an **atomic clock**, an extremely accurate measurement of time is made using a well-defined transition in the $^{133}\boldsymbol{Cs\ atom}$ (f = 9,192,631,770 Hz).
- In 1971 two American physicists, J. C. Hafele and Richard E. Keating, used **four cesium beam atomic clocks** to test the **time dilation effect**.
- They **flew the four portable cesium clocks** eastward and westward on regularly scheduled commercial jet airplanes around the world and **compared the time with a reference atomic time scale at rest** at the U.S. Naval Observatory in Washington, D.C.


AP/Wide World Photos.

Joseph Hafele and Richard Keating are shown unloading one of their atomic clocks and the associated electronics from an airplane in Tel Aviv, Israel, during a stopover in November 1971 on their round-the-world trip to test special relativity.



Two airplanes took off (at different times) from Washington, D.C., where the U.S. Naval Observatory is located. The airplanes traveled east and west around Earth as it rotated. Atomic clocks on the airplanes were compared with similar clocks kept at the observatory to show that the moving clocks in the airplanes ran slower.

Atomic Clock Measurement

- The trip **eastward** took 65.4 hours with 41.2 flight hours, whereas the **westward** trip, taken a week later, took 80.3 hours with 48.6 flight hours.
- **The comparison with the special theory of relativity is complicated by the rotation of the Earth and by a gravitational effect arising from the general theory of relativity.**
- The actual **relativistic predictions** and **experimental observations** for the time differences are

Travel	Predicted	Observed
Eastward	- 40 ± 23 ns	- 59 ± 10 ns
Westward	275 ± 21 ns	273 ± 7 ns

Atomic Clock Measurement

- A **negative time indicates** that the time on the moving clock is less than the reference clock.
- **The moving clocks lost time (ran slower) during the eastward trip, but gained time (ran faster) during the westward trip.**
- This occurs because of the **rotation of the Earth**, indicating that the flying clocks ticked faster or slower than the reference clocks on Earth.
- The special theory of relativity is **verified** within the experimental uncertainties.

Velocity Addition

- An interesting test of the **velocity addition relations** was made by T. Alväger and colleagues at the CERN nuclear and particle physics research facility on the border of Switzerland and France.
- They used a beam of almost $\mathbf{20 - \mathit{GeV}}$ $(\mathbf{20\ \mathit{x}\ 10^{9}\ \mathit{eV}})$ $\boldsymbol{protons}$ to strike a target to produce **neutral pions** (π^0) having energies of more than 6 GeV.
- The π^0 $(\beta \approx 0.99975)$ have a **very short half-life and soon decay into two γ rays**.
- In the rest frame of the π^0 the two $\gamma\ rays$ go off **in opposite directions**.
- The experimenters measured the velocity of the **γ rays** going in the forward direction in the laboratory.

Velocity Addition

- The **Galilean addition of velocities** would require the velocity of the γ rays to be $\boldsymbol{u = 0.99975c + c = 1.99975c}$, because the velocity of $\gamma\, rays$ is already c.
- However, the **relativistic velocity addition**, in which $\boldsymbol{v = 0.99975c}$ is the velocity of the π^0 rest frame with respect to the laboratory and $u' = c$ is the velocity of the $\gamma\, rays$ in the rest frame of the π^0, predicts the **velocity $\boldsymbol{u}$ of the $\boldsymbol{\gamma\, rays}$ measured in the laboratory** to be

$$u = \frac{c + 0.99975c}{1 + \dfrac{(0.99975c)(c)}{c^2}} = c$$

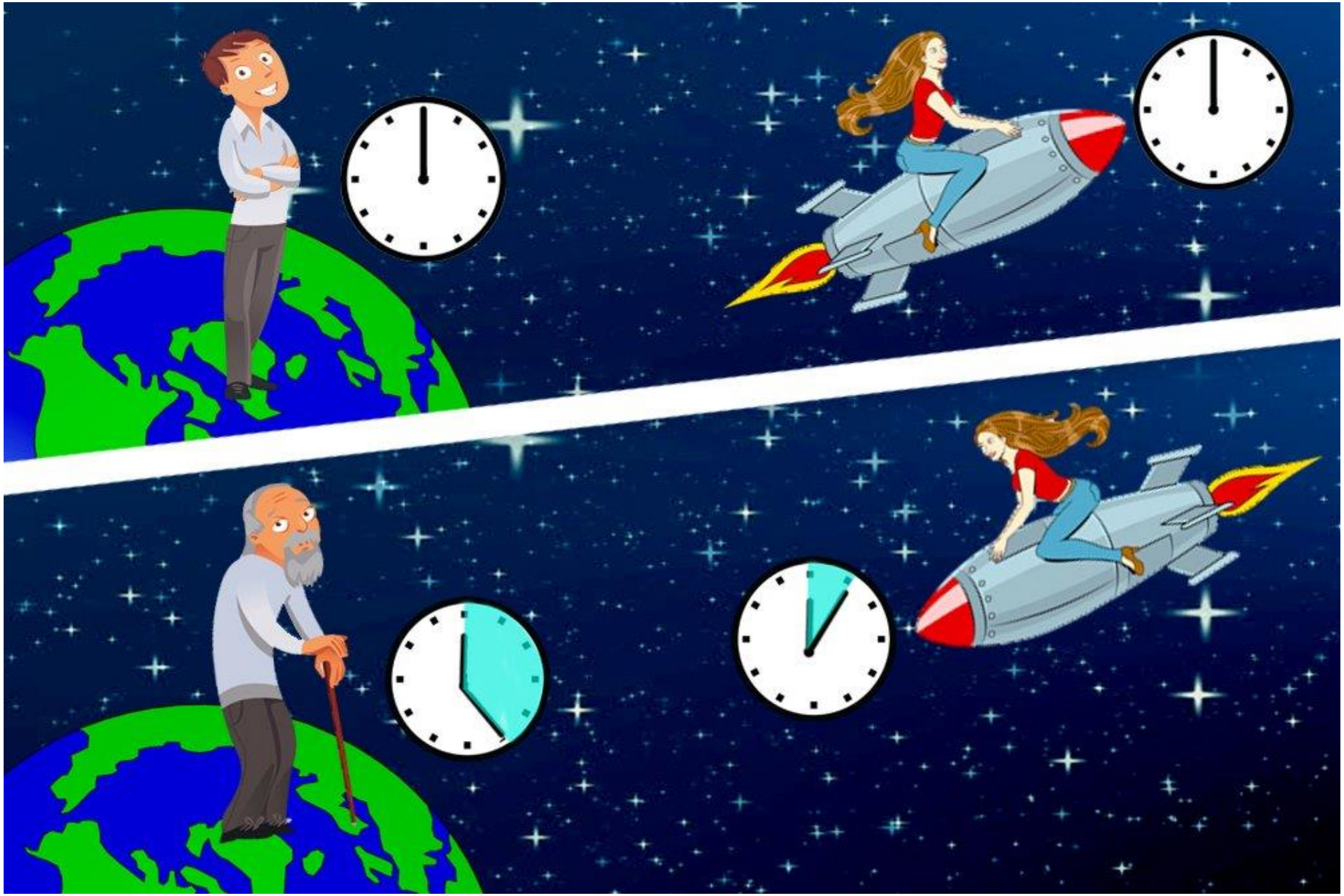
- The **experimental measurement** was accomplished by measuring the time taken for the $\gamma\, rays$ to travel between two detectors placed about **30 m apart** and was in excellent agreement with the relativistic prediction, **<u>but not the Galilean one</u>**.
- We again have conclusive **evidence of the need for the special theory of relativity**.

Testing Lorentz Symmetry

- Although we have mentioned only **three** rather interesting experiments, physicists performing experiments with nuclear and particle accelerators have examined thousands of cases that **verify the correctness** of the concepts discussed here.
- Quantum electrodynamics (QED) includes special relativity in its framework, and QED has been tested to one part in 10^{12}.
- **Lorentz symmetry requires the laws of physics to be the same for all observers, and Lorentz symmetry is important at the very foundation of our description of fundamental particles and forces.**
- Lorentz symmetry, together with the principles of quantum mechanics that are discussed in much of the remainder of this book, form the framework of relativistic quantum field theory.
- In just the past two decades, physicists have conceived and performed many experiments that test Lorentz symmetry, but no **violations have been discovered** to date.

Twin Paradox

- One of the most interesting topics in relativity is the **twin (or clock) paradox**.
- Almost from the time of publication of Einstein's famous paper in 1905, this subject has received considerable attention, and many variations exist.
- Suppose twins, **Mary and Frank**, choose different career paths.
- Mary (the **M**oving twin) becomes an astronaut and Frank (the **F**ixed twin) a stock broker.
- At age 30, Mary sets out on a spaceship to study a star system 8 ly from Earth.
- Mary **travels at very high speeds** to reach the star and returns during her life span.
- According to Frank's understanding of special relativity, **Mary's biological clock ticks more slowly** than his own, so he claims that *Mary will return from her trip younger* than he.



Twin Paradox

- **Mary** returns from her space journey as the **younger** twin.
- According to **Frank**, Mary's spaceship takes off from Earth and quickly reaches its travel speed of $\mathbf{0.8}\boldsymbol{c}$.
- She travels the distance of 8 ly to the star system, slows down and turns around quickly, and returns to Earth at the same speed.
- The **accelerations** (positive and negative) take **negligible times** compared to the travel times between Earth and the star system.
- **According to Frank**, Mary's travel time to the star is **10 years** [(8 ly)/0.8*c* = 10 y] and the return is also **10 years**, for a total travel time of 20 years, so that Frank will be 30 + 10 + 10 y = **50 years old** when Mary returns.
- However, because Mary's clock is ticking more slowly, her travel time to the star is only $10\sqrt{1-0.8^2}y=$ **6 years**. **Frank calculates that Mary will only be 30 + 6 + 6 y = 42 years old** when she **returns** with respect to his own clock at rest.

Twin Paradox

- The important fact here is that Frank’s clock **is in an inertial system during the entire trip**; however, **Mary’s** clock is **not**.
- As long as Mary is traveling at constant speed away from Frank, **both of them** can argue that **the other twin is aging less rapidly**.
- However, when **Mary slows down** to turn around, she leaves her original inertial system and eventually returns in a completely **different inertial system**.
- Mary’s claim is no longer valid, because she does not remain in the same inertial system.
- There is also no doubt as to who is in the inertial system.
- **Frank feels no acceleration** during Mary’s entire trip, but **Mary will definitely feel acceleration** during her reversal time, just as we do when we step hard on the **brakes of a car**.
- The acceleration at the beginning and the deceleration at the end of her trip present little problem, because the fixed and moving clocks could be compared if Mary were just passing by Frank each way.
- **It is Mary’s acceleration at the star system that is the key.**
- A careful analysis of Mary’s entire trip using special relativity, including acceleration, will be in agreement **with Frank’s assessment that Mary is younger**.

Twin Paradox

- Table analyzes the twin paradox.
- Both Mary and Frank send out signals at a frequency f (as measured by their own clock).
- We include in the table the **various journey timemarks and signals received during the trip,** with one column for the twin Frank who stayed at home and one for the astronaut twin Mary who went on the trip.
- Let the **total time of the trip** as measured on Earth be $\boldsymbol{T}$.
- The **speed of Mary's spaceship is** $\boldsymbol{v}$ (as measured on Earth), which gives a relativistic $factor\ \gamma$.
- The **distance Mary's spaceship goes** before turning around (as measured on Earth) is $\boldsymbol{L}$.

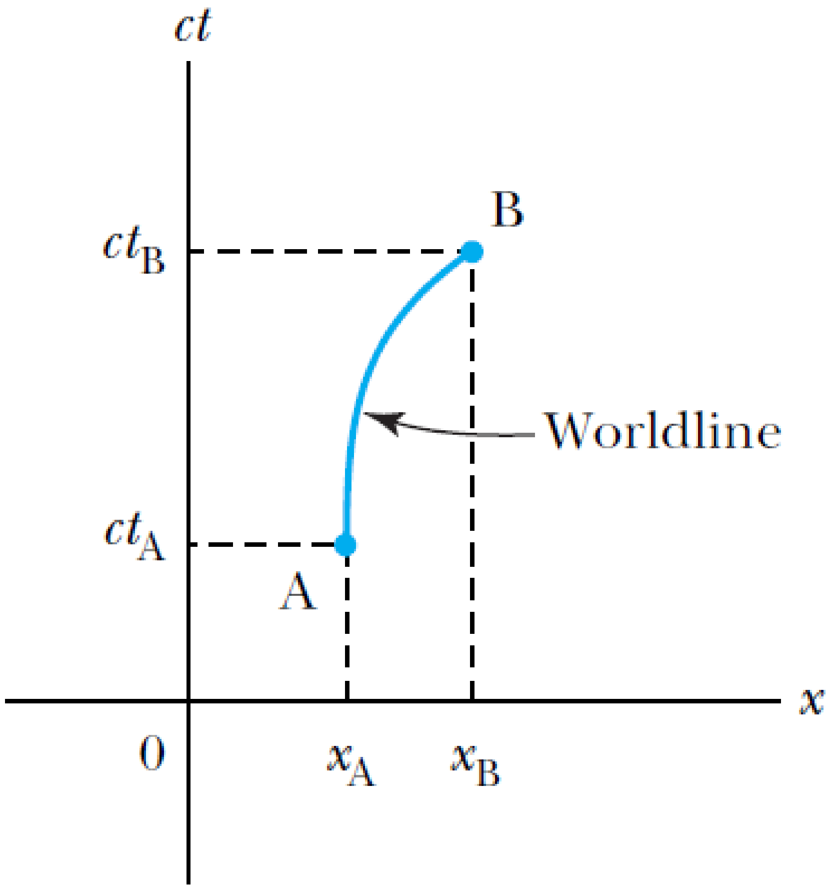
Table 2.1 Twin Paradox Analysis

Item	Measured by Frank (remains on Earth)	Measured by Mary (traveling astronaut)
Time of total trip	$T = 2L/v$	$T' = 2L/\gamma v$
Total number of signals sent	$fT = 2fL/v$	$fT' = 2fL/\gamma v$
Frequency of signals received at beginning of trip f'	$f\sqrt{\frac{1-\beta}{1+\beta}}$	$f\sqrt{\frac{1-\beta}{1+\beta}}$
Time of detecting Mary's turnaround	$t_1 = L/v + L/c$	$t_1' = L/\gamma v$
Number of signals received at the rate f'	$f't_1 = \frac{fL}{v}\sqrt{1-\beta^2}$	$f't_1' = \frac{fL}{v}(1-\beta)$
Time for remainder of trip	$t_2 = L/v - L/c$	$t_2' = L/\gamma v$
Frequency of signals received at end of trip f''	$f\sqrt{\frac{1+\beta}{1-\beta}}$	$f\sqrt{\frac{1+\beta}{1-\beta}}$
Number of signals received at rate f''	$f''t_2 = \frac{fL}{v}\sqrt{1-\beta^2}$	$f''t_2' = \frac{fL}{v}(1+\beta)$
Total number of signals received	$2fL/\gamma v$	$2fL/v$
Conclusion as to other twin's measure of time taken	$T' = 2L/\gamma v$	$T = 2L/v$

After A. French, *Special Relativity,* New York: Norton (1968), p. 158.

Spacetime

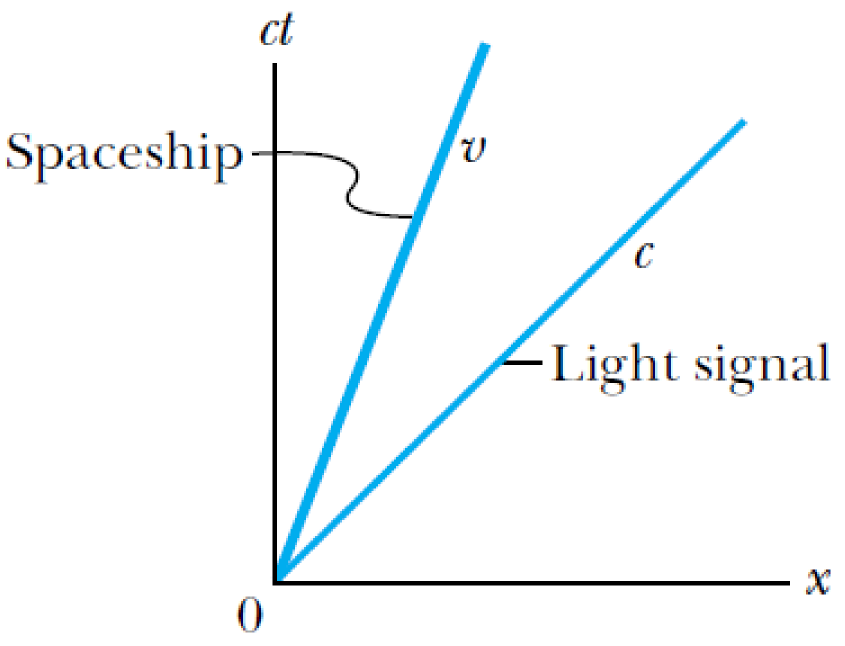
- When describing events in relativity, it is sometimes convenient to represent events on a **spacetime diagram**.
- For convenience we use only one spatial coordinate x and specify position in this one dimension.
- We use ct instead of time so that **both coordinates** will have **dimensions of length**.
- Spacetime diagrams were first used by H. Minkowski in 1908 and are often called **Minkowski diagrams**.
- We have learned in relativity that **we must denote both space and time to specify an event.**
- **This is the origin of the term *fourth dimension* for time**.
- The events for A and B in Figure are denoted by the respective coordinates (x_A, ct_A) and (x_B, ct_B), respectively.
- The line connecting events A and B is the path from A to B and is called a **worldline**.



- A spacetime diagram is used to specify events.
- The worldline denoting the path from event A to event B is shown.

Spacetime

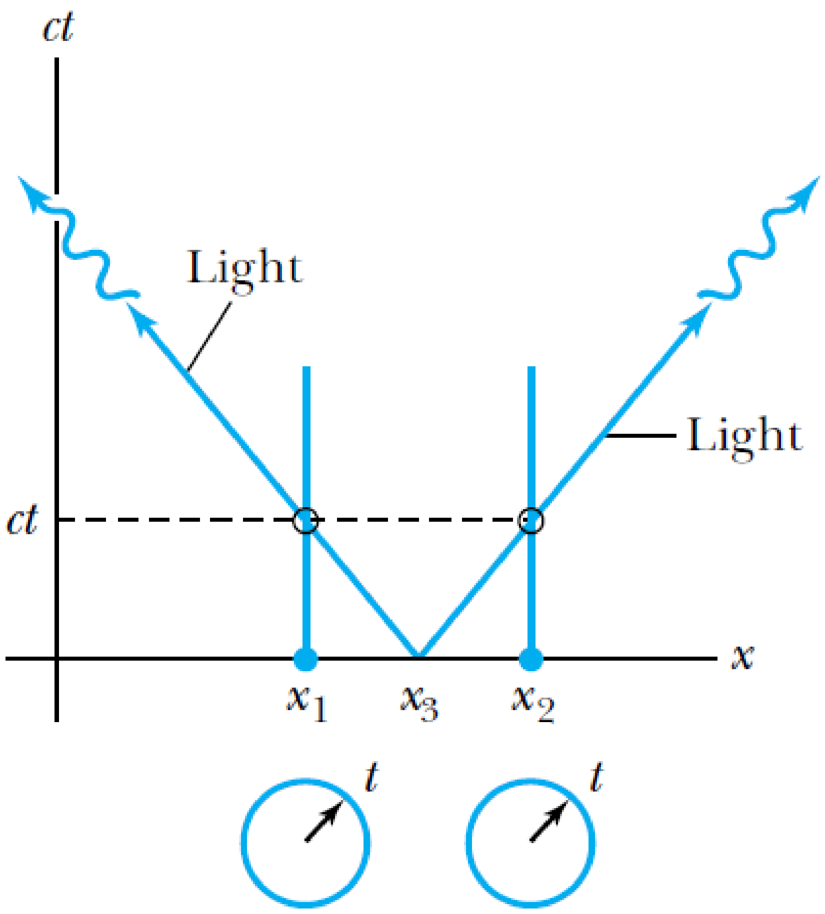
- A **spaceship** launched from $x = 0, ct = 0$ with constant velocity $\boldsymbol{v}$ has the worldline shown in Figure : a straight line with slope *c/v*.
- For example, a **light signal** sent out from the origin with speed $\boldsymbol{c}$ is represented on a spacetime graph with a worldline that has a slope $c/c = 1$, so that line makes an angle of **45°** with both the x and ct axes.
- Any real motion in the spacetime diagram cannot have a **slope of less than 1** because that motion would have a speed greater than *c.*
- The Lorentz transformation **does not** allow such a speed.



- A light signal has the slope of 45° on a spacetime diagram.
- A spaceship moving along the *x* axis with speed *v* is a straight line on the spacetime diagram with a slope *c/v*.

Spacetime

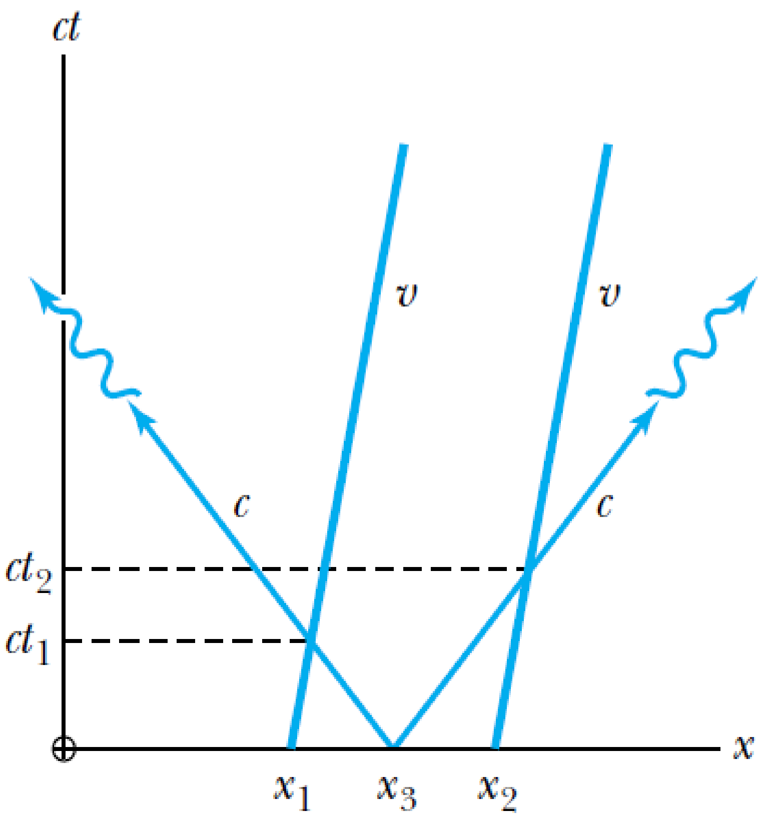
- Consider **two events** that occur at the same time ($ct = 0$) but at different positions, x_1 and x_2.
- We denote the events (x, ct) as **(x_1, 0) and (x_2, 0),** and we show them in Figure in an inertial system with an origin fixed at $x = 0$ and $ct = 0$.
- We must first devise a **method** that will allow us to determine experimentally that the events occurred **simultaneously.**
- Let us place **clocks** at positions x_1 and x_2 and place a flashbulb at position $\boldsymbol{x_3}$ **halfway** between x_1 and x_2.
- The two clocks have been previously **synchronized** and keep identical time.
- At time $t = 0$, the **flashbulb explodes** and sends out light signals from position x_3.
- The light signals proceed along their worldlines as shown in Figure.
- The two light signals arrive at positions x_1 and x_2 at **identical times** t as shown on the spacetime diagram.
- **By using such techniques we can be sure that events occur simultaneously in our inertial reference system.**



- Clocks positioned at x_1 and x_2 can be synchronized by sending a light signal from a position x_3 halfway between.
- The light signals intercept the worldlines of x_1 and x_2 at the same time t.

Spacetime

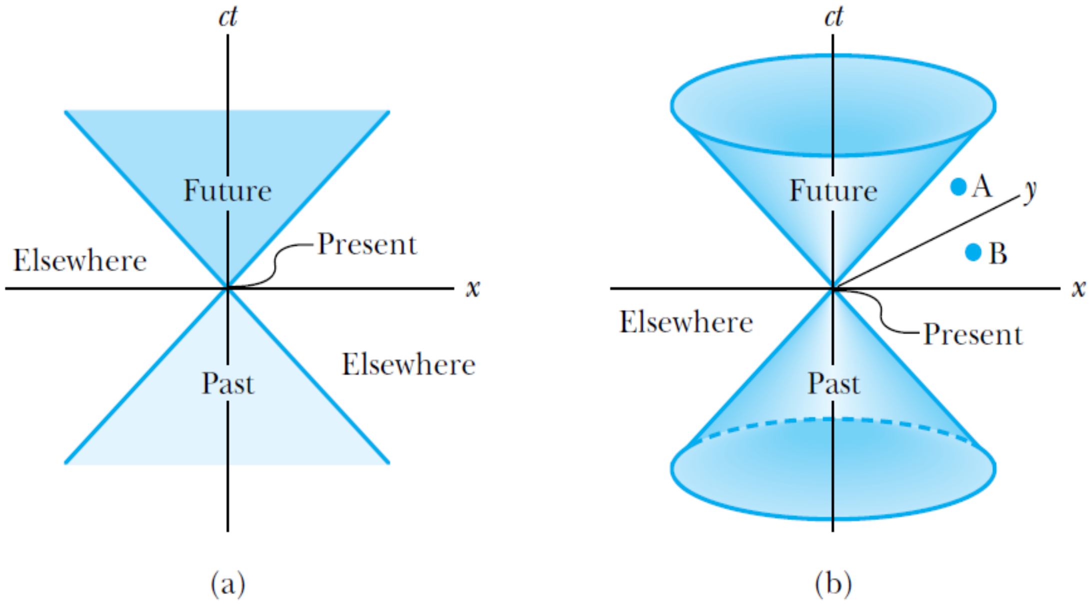
- **But what about other inertial reference systems**?
- We realize that the **two events** will not be simultaneous in a reference system K' moving at speed ***v*** with respect to our (x, ct) system.
- Because the **two events have different spatial coordinates**, x_1 and x_2, the Lorentz transformation will preclude them from occurring at the same time t' simultaneously in the moving coordinate systems.
- **We can see this by supposing that events 1, 2, and 3 take place on a spaceship moving with velocity *v*.**
- The **worldlines** for x_1 and x_2 are the **two slanted lines** beginning at x_1 and x_2 in Figure.
- However, when the flashbulb goes off, the light signals from x_3 still proceed at 45° in the (x, ct) reference system.
- **The light signals intersect the worldlines from positions x_1 and x_2 at different times**, so we do not see the events as being simultaneous in the moving system.
- Spacetime diagrams can be useful in showing such phenomena.



If the positions x_1 ($=x'_1$) and x_2 ($=x'_2$) of the previous figure are on a moving system K' when the flashbulb goes off, the times will not appear simultaneously in system K, because the worldlines for x'_1 and x'_2 are slanted.

Spacetime

- Anything that happened <u>earlier</u> in time than ***t* = 0 is called the *past*** and anything that occurs <u>after</u> ***t* = 0 is called the *future***.
- The spacetime diagram in Figure shows both the past and the future.
- Notice that only the events within the shaded area **below *t* = 0 can affect the present**.
- Events outside this area cannot affect the present because of the limitation $\boldsymbol{v} \leq \boldsymbol{c}$; this region is called ***elsewhere.***
- Similarly, **the present cannot affect any events occurring outside the shaded area above *t* = 0, again because of the limitation of the speed of light.**



(a)The spacetime diagram can be used to show the past, present, and future. Only causal events are placed inside the shaded area. **Events outside the shaded area below *t* = 0 cannot affect the present.**
(b) If we add an additional spatial coordinate ***y***, a space cone can be drawn. **The present cannot affect event A, but event B can.**

Spacetime

- If we add another spatial coordinate *y* to our spacetime coordinates, we will have a cone as shown in Figure b, which we refer to as the **light cone**.
- All causal events related to the present (x = 0, ct = 0) must be **within the light cone.**
- In Figure b, anything **occurring at present** (x = 0, ct = 0) **cannot possibly affect an event at position A**; however, the **event B can easily affect event A** because **A would be within the range of light signals emanating from B.**

Spacetime

- **Invariant quantities have the same value in all inertial frames**.
- They serve a special role in physics because their **values do not change** from one system to another.
- For example, **the speed of light *c* is invariant**.
- We are used to defining distances by $\boldsymbol{d^2 = x^2 + y^2 + z^2}$, and in **Euclidean geometry**, we obtain the same result for d^2 in any inertial frame of reference.
- If we refer to $\boldsymbol{x^2 + y^2 + z^2 = c^2t^2}$ we have similar equations in both systems K and K'. Let us look more carefully at the quantity s^2 defined as

$$s^2 = x^2 - (ct)^2$$

and also

$$s'^2 = x'^2 - (ct')^2$$

Spacetime

- If we use the Lorentz transformation for *x* and *t*, we find that $s^2 = s'^2$, so **s² is an invariant quantity**. This relationship can be extended to include the two other spatial coordinates, *y* and *z*, so that

$$s^2 = x^2 + y^2 + z^2 - (ct)^2$$

- **For simplicity**, we will sometimes continue to use only the single spatial coordinate ***x***. If we consider two events, we can determine the quantity **Δ*s*²** where

$$\Delta s^2 = \Delta x^2 - c^2 \Delta t^2$$

- between the two events, and we find that it is **invariant in any inertial frame**. The quantity **Δ*s*** is known as the **spacetime interval** between two events.

- There are three possibilities for the invariant quantity Δs^2.

1. ***Δs^2* = 0:** In this case $\Delta x^2 = c^2\Delta t^2$, and the two events can be connected only by a light signal. The events are said to have a **lightlike** separation.

2. ***Δs^2* > 0:** Here we must have $\Delta x^2 > c^2\Delta t^2$, and **no signal can travel fast enough** to connect the two events. The events are not causally connected and are said to have a **spacelike** separation. In this case we can always find an inertial frame **traveling at a velocity less than *c*** in which the two events can occur simultaneously in time but at different places in space.

3. ***Δs^2* < 0:** Here we have $\Delta x^2 < c^2\Delta t^2$, and the two events can be causally connected. The interval is said to be **timelike**. In this case we can find an inertial frame **traveling at a velocity less than *c*** in which the two events occur at the same position in space but at different times. The two events can never occur simultaneously.

Spacetime

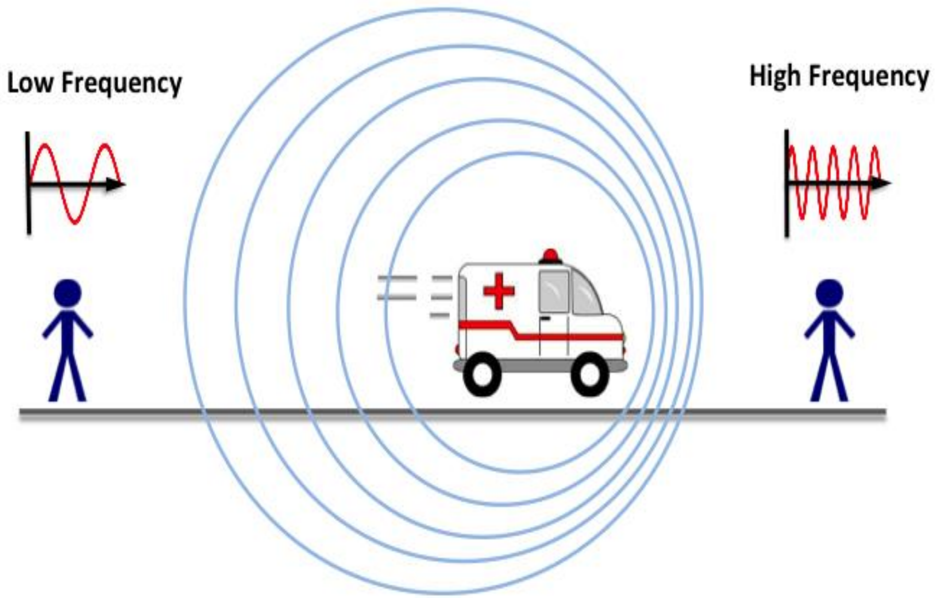
- A 3-vector $\vec{R}$ can be defined using Cartesian coordinates *x, y, z* in threedimensional Euclidean space.
- Another 3-vector $\vec{R}'$ can be determined in another Cartesian coordinate system using *x′, y′, z′* in the new system.
- So far in introductory physics we have discussed translations and rotations of axes between these two systems.
- We have learned that there are two geometries in Newtonian spacetime.
- One is the three-dimensional **Euclidean geometry** in which the space interval is $dl^2 = dx^2 + dy^2 + dz^2$, and the other is a one-dimensional **time interval *dt***.
- Minkowski pointed out that both space and time by themselves will not suffice under a Lorentz transformation, and only a union of both will be independent and useful.
- We can form a four-dimensional space or four-vector using the four components ***x, y, z, ict***.
- The Equation becomes

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$
$$ds'^2 = dx'^2 + dy'^2 + dz'^2 - c^2dt'^2$$
$$ds^2 = ds'^2$$

Spacetime

- We previously noted that ds^2 (actually Δs^2) can be **positive, negative, or zero**.
- With the four-vector formalism we only have the *spacetime* geometry, not separate geometries for space and time.
- **The spacetime distances $ds^2 = ds'^2$ are invariant under the Lorentz transformation**.
- We will learn how the **energy and momentum of a particle are connected**.
- Similar to the spacetime four-vector, there is an energy-momentum four-vector, and the invariant quantity is the mass.
- The four-vector formalism gives us equations that produce form-invariant quantities under appropriate Lorentz transformations.

Doppler Effect



- You may have already studied the Doppler effect of sound in introductory physics.
- **It causes an increased frequency of sound as a source such as a train (with whistle blowing) approaches a receiver (our eardrum) and a decrease in frequency as the source recedes.**
- In example you see an ambulance.

- A change in sound frequency also occurs when the **source is fixed** and the **receiver is moving**.
- The **change in frequency of the sound wave** depends on whether the source or receiver is moving.
- On first thought it seems that the Doppler effect in sound **violates the principle of relativity**, until we realize that there is in fact a special frame for sound waves.
- Sound waves depend on media such as **air, water, or a steel plate** to propagate.
- **For light, however, there is no such medium.**
- It is only relative motion of the source and receiver that is relevant, and we expect some differences between the **relativistic Doppler effect for light waves** and the **normal Doppler effect for sound**.

Doppler Effect

- It is not possible **for a source of light to travel faster than light in a vacuum**, but it is possible for a **source of sound** to travel **faster than the speed of sound**.
- Consider **a source of light** (for example, a star) **and a receiver** (an astronomer) approaching one another with a relative velocity v.
- First we consider the **receiver fixed** in system K and the light source in system K' moving toward the receiver with velocity $\boldsymbol{v}$.
- **The source emits *n* waves during the time interval *T*.**
- Because the speed of light is always *c* and the source is moving with velocity *v*, the total distance between the front and rear of the wave train emitted during the time interval *T* is

$$\boldsymbol{Lenght\ of\ wave\ train = cT - vT}$$

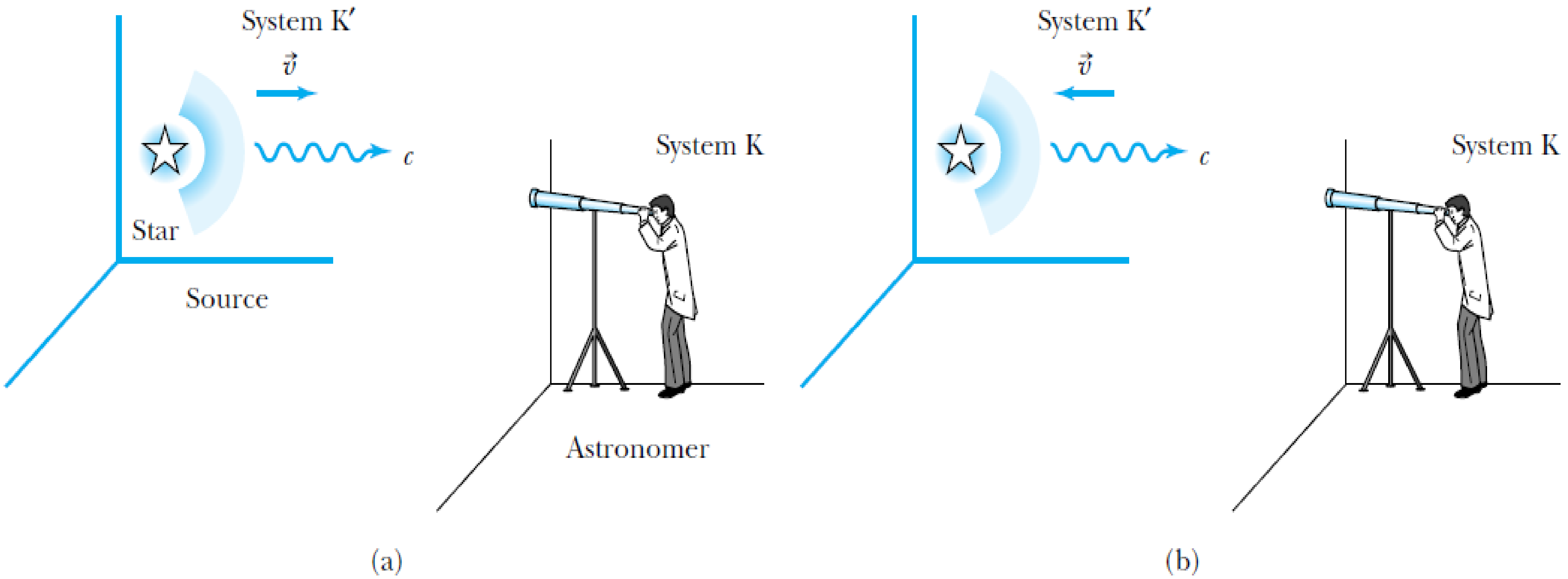
- Because there are ***n*** waves emitted during this time period, the **wavelength** must be

$$\lambda = \frac{cT - vT}{n}$$

and the **frequency**, $f = c/\lambda$, is

$$f = \frac{cn}{cT - vT}$$

Doppler Effect



(a) The source (star) is approaching the receiver (astronomer) with velocity v while it emits starlight signals with speed ***c***.

(b) Here the source and receiver are receding with velocity v. **The Doppler effect for light is different than that for sound, because of relativity and no medium to carry the light waves.**

Doppler Effect

- In its rest frame, **the source emits *n* waves of frequency f_0 during the proper time T'_0.**

$$n = f_0 T'_0$$

- The proper time interval T'_0 measured on the clock at rest in the moving system is related to the **time interval *T*** measured on a clock fixed by the receiver in system K by

$$T'_0 = \frac{T}{\gamma}$$

- where **γ is the relativistic factor**. **The clock moving with the source measures the proper time because it is present with both the beginning and end of the wave.**
- The **frequency** can be determined

$$f = \frac{c f_0 T/\gamma}{cT - vT} = \frac{1}{1 - v/c}\frac{f_0}{\gamma} = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} f_0$$

Doppler Effect

- where we have inserted the equation for γ. If we use $\boldsymbol{\beta} = \boldsymbol{v}/\boldsymbol{c}$, we can write the previous equation as

$$f = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0 \qquad \boldsymbol{Source\ and\ receiver\ approaching}$$

- It is straightforward to show that this Equation is also **valid** when the **source is fixed** and the **receiver approaches** it with velocity v.
- It is the relative velocity $\boldsymbol{v}$, of course, that is important. But what happens if the source and receiver are **receding** from each other with velocity v (see Figure b)? The derivation is similar to the one just done, except that the **distance between the beginning and end of the wave train becomes**

$$\boldsymbol{Lenght\ of\ wave\ train = cT + vT}$$

- because the source and receiver are **receding rather than approaching**.

Doppler Effect

- This change in sign is propagated throughout the derivation with the final result

$$f = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} f_0 \qquad \boldsymbol{Source\ and\ receiver\ receding}$$

- **These two equations can be combined into one equation** if we agree to use a + sign for β (+*v*/*c*) when the source and receiver are approaching each other and a - sign for β (-*v*/*c*) when they are receding.

$$f = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0 \qquad \boldsymbol{Relativistic\ Doppler\ Effect}$$

- The Doppler effect is useful in many areas of science including astronomy, atomic physics, and nuclear physics.

Doppler Effect

- Elements **absorb and emit** characteristic frequencies of light due to the existence of particular atomic levels.
- Scientists have observed these characteristic frequencies in starlight and have observed **shifts** in the frequencies.
- One reason for these **shifts** is the **Doppler effect**, and the frequency changes are used to determine the speed of the emitting object with respect to us.
- This is the source of the **redshifts** of starlight caused by objects moving away from us.
- These data have been used to ascertain that the **universe is expanding**.
- **The farther away the star, the higher the redshift. This observation is what led Harlow Shapley and Edwin Hubble to the idea that the universe started with a Big Bang.**

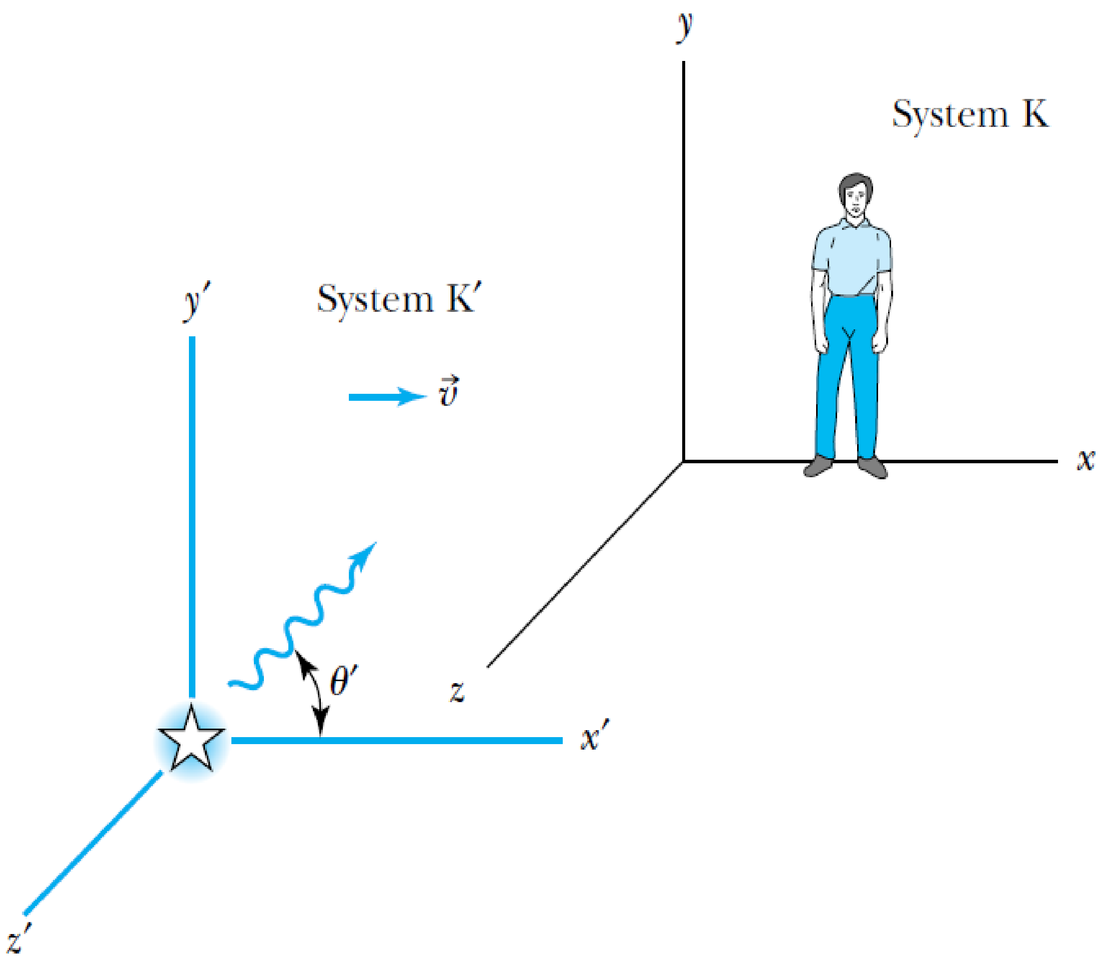
Doppler Effect

- We have only considered the **source** and **receiver to be directly approaching or receding**.
- It is also **possible** for the **two to be moving at an angle with respect to one another**, as shown in Figure.
- The angles θ and θ’ are the angles the light signals make with the ***x* axes** in the K and K’ systems. They are related by

$$f\cos\theta = \frac{f_0\,(\cos\theta' + \beta)}{\sqrt{1-\beta^2}} \qquad and \qquad f\sin\theta = f_0\sin\theta'$$

- The generalized **Doppler shift equation** becomes

$$f = \frac{1+\beta\cos\theta'}{\sqrt{1-\beta^2}}f_0$$



The light signals in system K’ are emitted at an angle θ’ from the *x’* axis and remain in the *x’*y’ plane.

Doppler Effect

- Note that when **$\theta' = 0°$** (source and receiver approaching) and when **$\theta' = 180°$** (source and receiver receding). This situation is known as the ***longitudinal Doppler effect***.

- When **$\theta' = 90°$** the emission is purely transverse to the direction of motion, and we have the ***transverse Doppler effect***, which is purely a **relativistic effect** that does not occur classically.

- The transverse Doppler effect is directly due to time dilation and has been verified experimentally.

Applications of the Doppler Effect

- ***Astronomy :*** Perhaps the best-known application is in astronomy, where the Doppler shifts of known atomic transition frequencies determine the relative velocities of astronomical objects with respect to us. Such measurements continue to be used today to find the distances of such unusual objects as quasars (objects having incredibly large masses that produce tremendous amounts of radiation; see Chapter 16). The Doppler effect has been used to discover other effects in astronomy, for example, the rate of rotation of Venus and the fact that Venus rotates in the opposite direction of Earth—the sun rises in the west on Venus. This was determined by observing light reflected from both sides of Venus—on one side it is blueshifted and on the other side it is redshifted, as shown in Figure A. The same technique has been used to determine the rate of rotation of stars.

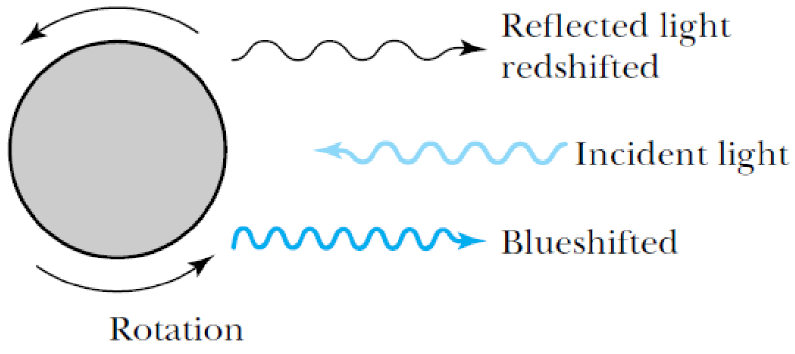


Figure A

Applications of the Doppler Effect

- ***Radar :*** The Doppler effect is nowhere more important than it is in radar. When an electromagnetic radar signal reflects off of a moving target, the so-called *echo* signal will be shifted in frequency by the Doppler effect. Very small frequency shifts can be determined by examining the beat frequency of the echo signal with a reference signal. The frequency shift is proportional to the radial component of the target's velocity. Navigation radar is quite complex, and ingenious techniques have been devised to determine the target position and velocity using multiple radar beams. By using pulsed Doppler radar it is possible to separate moving targets from stationary targets, called clutter.
- Doppler radar is also extensively used in meteorology. Vertical motion of airdrafts, sizes and motion of raindrops, motion of thunderstorms, and detailed patterns of wind distribution have all been studied with Doppler radar.
- X rays and gamma rays emitted from moving atoms and nuclei have their frequencies shifted by the Doppler effect. Such phenomena tend to broaden radiation frequencies emitted by stationary atoms and nuclei and add to the natural spectral widths observed.

Applications of the Doppler Effect

- ***Laser Cooling :*** In order to perform fundamental measurements in atomic physics, it is useful to limit the effects of thermal motion and to isolate single atoms. A method taking advantage of the Doppler effect can slow down even neutral atoms and eventually isolate them. Atoms emitted from a hot oven will have a spread of velocities. If these atoms form a beam as shown in Figure B, a laser beam impinging on the atoms from the right can slow them down by transferring momentum.

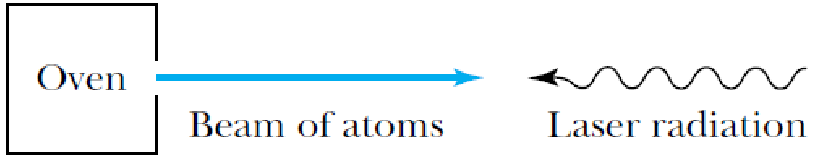


Figure B

- Atoms have characteristic energy levels that allow them to absorb and emit radiation of specific frequencies. Atoms moving with respect to the laser beam will "see" a shift in the laser frequency because of the Doppler effect. For example, atoms moving toward the laser beam will encounter light with high frequency, and atoms moving away from the laser beam will encounter light with low frequency. Even atoms moving in the same direction within the beam of atoms will see slightly different frequencies depending on the velocities of the various atoms. Now, if the frequency of the laser beam is tuned to the precise frequency seen by the faster atoms so that those atoms can be excited by absorbing the radiation, then those faster atoms will be slowed down by absorbing the momentum of the laser radiation. The slower atoms will "see" a laser beam that has been Doppler shifted to a lower frequency than is needed to absorb the radiation, and these atoms are not as likely to absorb the laser radiation. The net effect is that the atoms as a whole are *slowed down* and their *velocity spread is reduced*.

Applications of the Doppler Effect

- As the atoms slow down, they see that the Dopplershifted frequencies of the laser change, and the atoms no longer absorb the laser radiation. They continue with the same lower velocity and velocity spread. The lower temperature limits reached by Doppler cooling depend on the atom, but typical values are on the order of hundreds of microkelvins. Doppler cooling is normally accompanied by intersecting laser beams at different angles; an “optical molasses” can be created in which atoms are essentially trapped. Further cooling is obtained by other techniques including “Sisyphus” and evaporative cooling, among others. In a remarkable series of experiments by various researchers, atoms have been cooled to temperatures approaching 10^{-10} K. The 1997 Nobel Prize in Physics was awarded to Steven Chu, Claude Cohen-Tannoudji, and William Phillips for these techniques. An important use of laser cooling is for atomic clocks. See http://www.nist.gov/physlab/div847/grp50/primary-frequency-standards.cfm for a good discussion. See also Steven Chu, “Laser Trapping of Neutral Particles,” *Scientific American* 266, 70 (February 1992). In Chapter 9 we will discuss how laser cooling is used to produce an ultracold state of matter known as a Bose-Einstein condensate.